

Robust \mathcal{H}_∞ Controller Synthesis for Hovering Ducted-Fan VTOL Micro-UAV

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Abstract—This paper models an experimental vertical take-off landing (VTOL) micro unmanned aerial vehicle (UAV) device developed at the University of Applied Sciences OST RJ in Rapperswil. The model is then converted to a linear time invariant (LTI) state space plant to design a robust controller using \mathcal{H}_∞ synthesis that tracks a piecewise constant spatial position reference. In conclusion simulated preliminary results on the controller performance are presented.

I. INTRODUCTION

The presented VTOL micro UAV is a device developed by T. Rothlin [1] at the University of Applied Sciences OST RJ, Rapperswil that can be steered only by controlling the propeller velocity and the angle of attack of four flaps under the ducted fan. Currently there is not a control algorithm for the UAV and the uncertainties from aerodynamics and gyroscopic effects caused by the propeller make this an interesting candidate for robust controller synthesis.

II. SYSTEM MODELLING

To model the dynamics of the ducted-fan UAV two reference frames are required: an inertial frame and a body-frame attached to the center of mass [2]. In the inertial frame we work in the base given by the unit vectors $\{\hat{i}, \hat{j}, \hat{k}\}$, whereas in the body-frame we use $\{\hat{x}, \hat{y}, \hat{z}\}$ together with the Euler angles $\Theta = [\phi \ \vartheta \ \psi]^T$. The angular velocity $\Omega = [p \ q \ r]^T$ is then related to Θ by $\dot{\Theta} = \mathbf{U}\Omega$. To rotate from the inertial frame to body frame we use the $SO(3)$ matrix \mathbf{R} . Both \mathbf{U} and \mathbf{R} can be found in [2].

A. Equations of Motion

Consider a simplified rigid-body physical model sketched in Fig. 1. With respect to the inertial frame the equations of motion from the Newton-Euler formalism are

$$m\ddot{\mathbf{P}} = \mathbf{R}^T \mathbf{F} \quad (1a)$$

$$\mathbf{J}\dot{\Omega} = -\Omega \times \mathbf{J}\Omega + \boldsymbol{\tau}, \quad (1b)$$

where m is the total mass, \mathbf{P} is the position, \mathbf{F} and $\boldsymbol{\tau}$ the total force and torque in the body frame respectively, and finally \mathbf{J} the moment of inertia, which is assumed to be a diagonal matrix.

In the body frame we model the total force \mathbf{F} acting on the UAV following [3] by considering a thrust force from the ducted-fan $\mathbf{F}_T = -k_T \omega^2 \hat{z}$ as a function of the propeller's

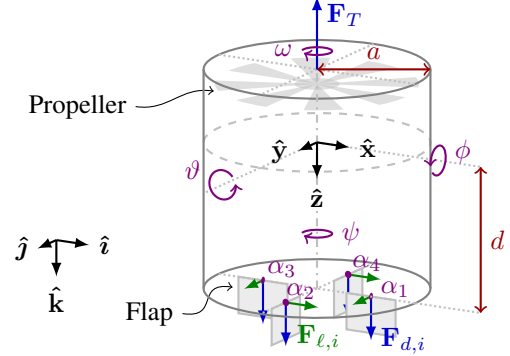


Fig. 1: Sketch of the rigid body physical model for the UAV.

angular velocity ω . Because of the geometry we approximate the generated air velocity field in the duct as being constant and collinear to \hat{z} [4]

$$\boldsymbol{\nu} = \sqrt{\frac{F_T}{2a\rho\pi^2}} \hat{z} = \frac{\omega}{\pi} \sqrt{\frac{k_T}{2a\rho}} \hat{z}. \quad (2)$$

The *drag* and *lift* forces generated by the interaction of each flap with the air velocity field are then $\mathbf{F}_d = \frac{1}{2}\rho S C_d \nu^2 \hat{z}$ and $\mathbf{F}_l = \frac{1}{2}\rho S C_l \nu^2 \hat{n}$ respectively (\hat{n} being given by the orientation of the flap). The density of air ρ is assumed to be constant, while the drag coefficients S , C_d and C_l depend on the angle of attack α of the flap making them quite difficult to determine. Hereinafter, under the assumption that α is small they will be approximated with $C_d = c_d \alpha^2 + c_0$, $C_l = c_l \alpha$ [2, 4] and S is considered a constant. Finally, gravity adds a term $\mathbf{F}_g = mg\mathbf{R}^T \hat{k}$.

For the total torque it is derived from the geometry that each flap induces $\boldsymbol{\tau}_f = (\frac{1}{3}a\hat{t} + d\hat{z}) \times (\mathbf{F}_d + \mathbf{F}_l)$, with $\hat{t} = \hat{n} \times \hat{z}$. In addition there is also a torque induced by the gyroscopic procession $\boldsymbol{\tau}_g = \mathbf{R}^T \omega J_r \hat{k} \times \Omega$ with J_r being the inertia of the propeller with respect to its spin axis [3]. The resulting total quantities are then

$$\mathbf{F} = mg\mathbf{R}^T \hat{k} - k_T \omega^2 \hat{z} + \frac{\rho S \nu^2}{2} \sum_i (c_d \alpha_i^2 + c_0) \hat{z} + c_l \alpha_i \hat{n}_i, \quad (3a)$$

$$\boldsymbol{\tau} = \omega J_r \mathbf{R}^T (\hat{k} \times \Omega) + \frac{\rho c_l S \nu^2 d}{2} \hat{z} \times \sum_i \alpha_i \hat{n}_i. \quad (3b)$$

B. Linearized Dynamics and Plant Model

To synthesize a controller we simplify the dynamics to a state-space LTI model $\dot{\mathbf{x}} = \mathbf{A}_x \mathbf{x} + \mathbf{B}_{xu} \mathbf{u}$, $\mathbf{y} = \mathbf{C}_{yx} \mathbf{x} + \mathbf{D}_{yu} \mathbf{u}$

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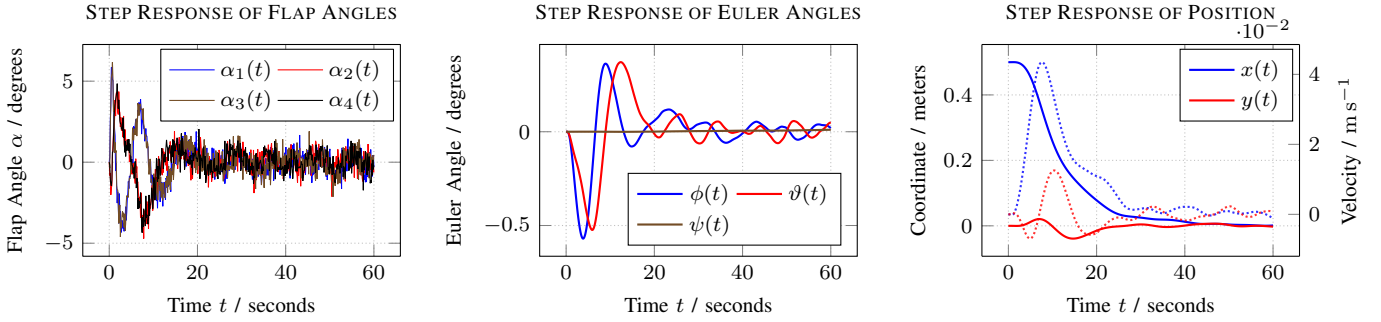


Fig. 2: Simulated step responses along x of the closed loop \mathcal{H}_∞ controller design. In the third plot from the left, continuous lines show the position error while dotted lines with the right axis are the velocity in the corresponding directions.

with state and inputs given by

$$\mathbf{x} = [\mathbf{P}^\top \quad \dot{\mathbf{P}}^\top \quad \Theta^\top \quad \Omega^\top]^\top, \quad \mathbf{u} = [\alpha^\top \quad \omega]^\top \quad (4)$$

respectively. The linearization is performed in a stationary hovering state at height h above the ground and yaw angle of 45° or $\mathbf{P}_0 = -h\hat{\mathbf{k}}$, $\Theta_0 = \frac{\pi}{4}\hat{\psi}$, thus $x_0 = [\mathbf{P}_0^\top \quad \mathbf{0}^\top \quad \Theta_0^\top \quad \mathbf{0}^\top]^\top$ and $u_0 = [\mathbf{0}^\top \quad \omega_0]^\top$, where $\omega_0 \approx \sqrt{mg/k_T}$ is the propeller's angular velocity to make the UAV hover.

The state is assumed to be known, as there is an inertial measurement unit with a dedicated sensor fusion chip onboard [1]. Thus, we only need to model a measurement delay T_m and the actuators. The linearized dynamics are extended with 2nd order Padé approximant $G_m(s) \approx e^{-sT_m}$ for the output delay and two transfer functions $G_\alpha(s)$, $G_\omega(s)$ for the flaps and thruster respectively. Specifically, $G_\alpha(s)$ is a 2nd order low pass filter (LPF) with a slight overshoot, while $G_\omega(s)$ is a first order LPF. All inputs and outputs are then normalized and a source of additive Gaussian noise $\mathbf{w}_\alpha \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ is introduced to model the wiggling in the servo motor of each flap. The resulting plant has 4 unobservable and 4 uncontrollable modes, however it remains both detectable and stabilizable.

III. CONTROLLER DESIGN

The control objective is to track a piecewise constant position reference $\mathbf{r}(t) \in \mathbb{R}^3$ in the inertial frame. The controller should move the UAV to $\mathbf{r}(t)$ with control actions that do not exceed the actuator limits $\alpha \in (\underline{\alpha}, \bar{\alpha})$, $\omega \in [0, \bar{\omega})$, do not bring the angular components of the state too far from their set point, i.e. $\Theta \in (-\varphi, \varphi)^3$, and do not exceed the velocity limits $\|\dot{\mathbf{P}}\|_1 \leq \bar{v}$.

To perform \mathcal{H}_∞ synthesis the system is first rewritten in the form presented in [5], in terms of control inputs $\mathbf{u} \in \mathbb{R}^5$, exogenous inputs $\mathbf{w} = [\mathbf{w}_\alpha^\top \quad \mathbf{r}^\top]^\top \in \mathbb{R}^7$, control outputs $\mathbf{y}_m = [(\mathbf{r} - \mathbf{y})^\top \quad \dot{\mathbf{P}}^\top \quad \Theta^\top \quad \Omega^\top]^\top$ and errors $\mathbf{e} = [\alpha^\top \quad \omega \quad (\mathbf{r} - \mathbf{y})^\top \quad \dot{\mathbf{P}}^\top \quad \Theta^\top]^\top$ and partitioned accordingly:

$$\begin{bmatrix} \mathbf{e} \\ \mathbf{y}_m \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{ew} & \mathbf{B}_{eu} \\ \mathbf{C}_{yw} & \mathbf{D}_{yu} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix} = \mathbf{G}_{\text{nom}} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix}. \quad (5)$$

Then for each component of the error a performance weight function is defined resulting in 5 transfer functions $W_{P,\alpha}(s)$, $W_{P,\omega}(s)$, $W_{P,\mathbf{P}}(s)$, $W_{P,\dot{\mathbf{P}}}(s)$, $W_{P,\Theta}(s)$. The \mathcal{H}_∞ synthesis is performed using the Riccati method yielding a stable controller

\mathbf{K}_∞ with $\gamma = \|\mathcal{L}(\mathbf{G}_{\text{nom}}, \mathbf{K}_\infty)\|_\infty \approx 0.9091$. Here we used the notation \mathcal{L} for the linear fractional transformation interconnection.

IV. PRELIMINARY RESULTS

The resulting closed loop plant $\mathbf{G}_{\text{cl}} = \mathcal{L}(\mathbf{G}_{\text{nom}}, \mathbf{K}_\infty)$ is simulated with a step of 50 cm along the x spatial coordinate and shown in Fig. 2. Currently, it takes the UAV about 30 seconds to track a half meter step, while respecting the constraints on the actuators. Although stable and with good noise rejection characteristics the current design does not take into account large inaccuracies caused by the highly non linear equations of motion. For higher speeds necessitate higher pitch and roll angles, a more performant controller will need to cope with higher modelling errors. Hence the next controller will be designed with μ -synthesis which can take into account the aforementioned modelling uncertainties.

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