Circularly-Symmetric Gaussian random vectors

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January 1, 2008

Abstract

A number of basic properties about circularly-symmetric Gaussian random vectors are stated and proved here. These properties are each probably well known to most researchers who work with Gaussian noise, but I have not found them stated together with simple proofs in the literature. They are usually viewed as too advanced or too detailed for elementary texts but are used (correctly or incorrectly) without discussion in more advanced work. These results should have appeared in Section 7.8.1 of my book, R. G. Gallager, "Principles of Digital Communication," Cambridge Press, 2008 (PDC08), but I came to understand them only while preparing a solution manual when the book was in the final production stage.

1 Pseudo-covariance and an example

Let $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_n)^{\mathsf{T}}$ be a complex jointly-Gaussian random vector. That is, $\Re(Z_k)$ and $\Im(Z_k)$ for $1 \le k \le n$ comprise a set of $2n$ jointly-Gaussian (real) random variables (rv's). For a large portion of the situations in which it is useful to view $2n$ jointly-Gaussian rv's as a vector of n complex jointly-Gaussian rv's, these vectors have an additional property called *circular symmetry*. By definition, $\boldsymbol{Z}$ is circularly symmetric if $e^{i\phi}\boldsymbol{Z}$ has the same probability distribution as $\boldsymbol{Z}$ for all real ϕ. For $n = 1$, *i.e.*, for the case where $\boldsymbol{Z}$ is a complex Gaussian random variable Z, circular symmetry holds if and only if $\Re(Z)$ and $\Im(Z)$ are statistically independent and identically distributed (iid) with zero mean, *i.e.*, if and only if $\Re(Z)$ and $\Im(Z)$ are jointly Gaussian with equi-probability-density contours around 0.

Since $\mathsf{E}[e^{i\phi}\boldsymbol{Z}] = e^{i\phi}\mathsf{E}[\boldsymbol{Z}]$, any circularly-symmetric complex random vector must have $\mathsf{E}[\boldsymbol{Z}] = 0$, *i.e.*, must have zero mean. In a moment, we will see that a circularly-symmetric jointly-Gaussian complex random vector is completely determined by its covariance matrix, $\mathsf{K}_{\boldsymbol{Z}} = \mathsf{E}[\boldsymbol{Z}\boldsymbol{Z}^{\dagger}]$, where $\boldsymbol{Z}^{\dagger} = \boldsymbol{Z}^{\mathsf{T}*}$ is the complex conjugate of the transpose. A circularly-symmetric jointly-Gaussian complex random vector $\boldsymbol{Z}$ is denoted and referred to as $\boldsymbol{Z} \sim \mathcal{CN}(0, \mathsf{K}_{\boldsymbol{Z}})$, where the $\mathcal{C}$ denotes that $\boldsymbol{Z}$ is both circularly symmetric and complex.

Most communication engineers believe that vectors of Gaussian random variables (real or complex) are determined by their covariance matrix. For the real case, this is only true when the variables are *jointly* Gaussian (see Section 7.8, PDC08). For the complex case, as emphasized and explained here, it is only true when the variables are both jointly Gaussian and circularly symmetric.

Example 1: Consider a vector $\boldsymbol{Z} = (Z_1, Z_2)^\mathsf{T}$ where $Z_1 \sim \mathcal{CN}(0, 1)$ and $Z_2 = Z_1^*$. Then Z_1 has iid real and imaginary parts, each $\mathcal{N}(0, 1/2)$. Z_2 also has iid real and imaginary parts, both Gaussian, so it is also circularly symmetric. The real and imaginary parts of Z_2 are jointly Gaussian with those of Z_1, so $\boldsymbol{Z}$ is jointly Gaussian with circularly-symmetric components. On the other hand, for Z_1 real (or approximately real), $Z_2 = Z_1$ (or $Z_2 \approx Z_1$). When Z_1 is pure imaginary (or close to pure imaginary), Z_2 is the negative of Z_1 (or $Z_2 \approx -Z_1$). Thus $\boldsymbol{Z}$ doesn't appear to have the required circular symmetry (we justify this more precisely later).

The covariance matrix for this example is easily calculated to be $\mathsf{K}_{\boldsymbol{Z}} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. This is also the covariance function of two iid unit variance complex rv's. The point of this example, then, is first that individual circular symmetry among the components of a random vector is not enough to provide overall circular symmetry, and second that complex jointly-Gaussian random vectors are not fully specified by their covariance matrices.

We have now seen that there is something wrong with the conventional wisdom and also that circular symmetry for random vectors is more subtle than circular symmetry for individual random variables. To make matters worse, we now show that the covariance matrix is *always* defective in determining circular symmetry. In particular, for any zero-mean complex random vector $\boldsymbol{Z}$, $\mathsf{K}_{e^{i\phi}\boldsymbol{Z}}$ is the same as $\mathsf{K}_{\boldsymbol{Z}}$. To see this,

$$\mathsf{K}_{e^{i\phi}\boldsymbol{Z}} = \mathsf{E}[(e^{i\phi}\boldsymbol{Z})(e^{-i\phi}\boldsymbol{Z}^*)^\mathsf{T}] = \mathsf{K}_{\boldsymbol{Z}}. \tag{1}$$

It is now time to dig ourselves out from this muddle of conventional wisdom. Although the covariance matrix *does* fully specify the distribution of a zero-mean real jointly-Gaussian random vector, there is another matrix, called the pseudo-covariance matrix, $\mathsf{M}_{\boldsymbol{Z}} = \mathsf{E}[\boldsymbol{Z}\boldsymbol{Z}^\mathsf{T}]$, that is needed, along with $\mathsf{K}_{\boldsymbol{Z}}$, to specify the distribution of an arbitrary complex jointly-Gaussian random vector. For the example above, $\mathsf{M}_{\boldsymbol{Z}} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, whereas for the case of two iid complex rv's, $\mathsf{M}_{\boldsymbol{Z}} = 0$.

The probability density of a complex random vector $\boldsymbol{Z} = (Z_1, \dots, Z_n)^\mathsf{T}$ is defined to be the probability density of the $2n$ real rv's $\boldsymbol{V} = (\Re(Z_1), \dots, \Re(Z_n), \Im(Z_1), \dots, \Im(Z_n))^\mathsf{T}$. If $\boldsymbol{Z}$ is zero-mean and complex jointly Gaussian, then $\boldsymbol{V}$ is determined by its $2n$ by $2n$ covariance matrix, say $\mathsf{K}_{\boldsymbol{V}}$. It turns out that $\mathsf{K}_{\boldsymbol{Z}}$ and $\mathsf{M}_{\boldsymbol{Z}}$ together specify $\mathsf{K}_{\boldsymbol{V}}$. In fact, by calculating the k, j element of $\mathsf{K}_{\boldsymbol{Z}}$ and $\mathsf{M}_{\boldsymbol{Z}}$, we see that

$$\mathsf{E}[\Re(Z_k)\Re(Z_j)] = \frac{1}{2}\left[\Re(\mathsf{K}_{\boldsymbol{Z}}(k,j)) + \Re(\mathsf{M}_{\boldsymbol{Z}}(k,j))\right], \tag{2}$$

$$\mathsf{E}[\Im(Z_k)\Im(Z_j)] = \frac{1}{2}\left[\Re(\mathsf{K}_{\boldsymbol{Z}}(k,j)) - \Re(\mathsf{M}_{\boldsymbol{Z}}(k,j))\right], \tag{3}$$

$$\mathsf{E}[\Re(Z_k)\Im(Z_j)] = \frac{1}{2}\left[-\Im(\mathsf{K}_{\boldsymbol{Z}}(k,j)) + \Im(\mathsf{M}_{\boldsymbol{Z}}(k,j))\right], \tag{4}$$

$$\mathsf{E}[\Im(Z_k)\Re(Z_j)] = \frac{1}{2}\left[\Im(\mathsf{K}_{\boldsymbol{Z}}(k,j)) + \Im(\mathsf{M}_{\boldsymbol{Z}}(k,j))\right]. \tag{5}$$

Thus the covariance of each pair of the $2n$ rv's, and thus the joint probability distribution of $\boldsymbol{Z}$, is determined by $\mathsf{K}_{\boldsymbol{Z}}$ and $\mathsf{M}_{\boldsymbol{Z}}$. In order for $\boldsymbol{Z}$ and $e^{i\phi}\boldsymbol{Z}$ to have the same distribution, it is necessary and sufficient to satisfy $\mathsf{K}_{e^{i\phi}\boldsymbol{Z}} = \mathsf{K}_{\boldsymbol{Z}}$ and $\mathsf{M}_{e^{i\phi}\boldsymbol{Z}} = \mathsf{M}_{\boldsymbol{Z}}$. The first condition is automatically satisfied by (1), so $\mathsf{M}_{\boldsymbol{Z}} = \mathsf{M}_{e^{i\phi}\boldsymbol{Z}}$ is necessary and sufficient for circular symmetry. On the other hand

$$\mathsf{M}_{e^{i\phi}\boldsymbol{Z}} = \mathsf{E}[(e^{i\phi}\boldsymbol{Z})(e^{i\phi}\boldsymbol{Z})^{\mathsf{T}}] = e^{2i\phi}\mathsf{M}_{\boldsymbol{Z}}$$

Thus[1] $\mathsf{M}_{e^{i\phi}\boldsymbol{Z}} = \mathsf{M}_{\boldsymbol{Z}}$ if and only if $\mathsf{M}_{\boldsymbol{Z}} = 0$.

This is summarized in the following theorem:

Theorem 1. *Assume that $\boldsymbol{Z}$ is a complex jointly-Gaussian random vector. Then $\boldsymbol{Z}$ is circularly symmetric if and only if $\mathsf{M}_{\boldsymbol{Z}} = 0$. In this case, the distribution of $\boldsymbol{Z}$ is determined by $\mathsf{K}_{\boldsymbol{Z}}$.*

This theorem explains the above example. In the example, $\mathsf{M}_{\boldsymbol{Z}} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and thus $\mathsf{M}_{e^{i\phi}\boldsymbol{Z}} = \begin{bmatrix} 0 & e^{2i\phi} \\ e^{2i\phi} & 0 \end{bmatrix}$. Thus, $\boldsymbol{Z}$ and $e^{i\phi}\boldsymbol{Z}$ do not have the same distribution even though, as complex random vectors, they have the same covariance matrix.

The definition of circular symmetry seems to capture the intuitive notion of circular symmetry somewhat better than the condition $\mathsf{M}_{\boldsymbol{Z}} = 0$, but the latter is often easier to work with.

Example 2: We now give an example illustrating the importance of the jointly-Gaussian requirement (as opposed to an individually Gaussian requirement). Consider a complex random 2-vector $\boldsymbol{Z}$ for which $Z_1 \sim \mathcal{CN}(0,1)$ and $Z_2 = UZ_1$ where U is statistically independent of Z_1 and has possible values ± 1 with probability 1/2 each. Then Z_1 and Z_2 are not jointly Gaussian, and in fact $\Re(Z_1)$ and $\Re(Z_2)$ have a joint distribution that is concentrated on the two diagonal axes. It is easy to see that $\mathsf{K}_{\boldsymbol{Z}} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\mathsf{M}_{\boldsymbol{Z}} = 0$ and thus $\boldsymbol{Z}$ is circularly symmetric. The corresponding real random 4-vector $\boldsymbol{V}$ satisfies $\mathsf{K}_{\boldsymbol{V}} = \mathsf{I}_4$, so the rv's are uncorrelated. However, they are clearly not independent and the probability density does not exist.

The point of this example is that the joint Gaussian property is important in the results that have been derived. The stubborn believer in an oversimplified world might try to think that Examples 1 and 2 rely on the lack of a probability density. This is not true, and each example would lead to the same conclusion if we modified $\boldsymbol{Z}$ by adding a nice random vector $\boldsymbol{Y} \sim \mathcal{CN}(0, \varepsilon)$. This would yield a true probability density and slightly complicate but not change the issues discussed.

[1]Note that the statement $\mathsf{M}_{e^{i\phi}\boldsymbol{Z}} = \mathsf{M}_{\boldsymbol{Z}}$ if and only if $\mathsf{M}_{\boldsymbol{Z}} = 0$ is valid whether or not $\boldsymbol{Z}$ is Gaussian. Similarly (1) is valid whether or not $\boldsymbol{Z}$ is Gaussian.

2 Linear transformations of circularly-symmetric Gaussian vectors

Circular symmetry can also be characterized as the result of a linear transformation (in $\mathbb{C}$) of m iid random variables, each $\mathcal{CN}(0,1)$. Thus let $\boldsymbol{W} \sim \mathcal{CN}(0, \mathsf{I}_m)$. That is, $W_1, \dots, W_m$ are iid and have iid real and imaginary parts, thus consisting collectively of $2n$ iid real rv's, each $\mathcal{N}(0, \frac{1}{2})$. Let A be an arbitrary m by n complex matrix and let $\boldsymbol{Z} = \mathsf{A}\boldsymbol{W}$. Then

$$\mathsf{K}_{\boldsymbol{Z}} = \mathsf{E}[\mathsf{A}\boldsymbol{W}\boldsymbol{W}^\dagger\mathsf{A}^\dagger] = \mathsf{A}\mathsf{A}^\dagger; \qquad \mathsf{M}_{\boldsymbol{Z}} = \mathsf{E}[\mathsf{A}\boldsymbol{W}\boldsymbol{W}^\mathsf{T}\mathsf{A}^\mathsf{T}] = 0$$

Thus $\boldsymbol{Z}$ is circularly symmetric and denoted by $\mathcal{CN}(0, \mathsf{A}\mathsf{A}^\dagger)$,

As explained in greater detail in Section 7.11.1 of PDC08, the class of covariance matrices (real or complex) is the same as the class of nonnegative-definite matrices. Each such matrix K can be represented as

$$\mathsf{K} = \mathsf{Q}\Lambda\mathsf{Q}^{-1} \tag{6}$$

where Λ is diagonal with nonnegative terms which are the eigenvalues of K. The columns of Q are orthonormal eigenvectors[2] of those eigenvalues. By choosing $\mathsf{R} = \mathsf{Q}\sqrt{\Lambda}\mathsf{Q}^{-1}$, we see that $\boldsymbol{Z} = \mathsf{R}\boldsymbol{W}$ has the covariance matrix $\mathsf{K}_{\boldsymbol{Z}} = \mathsf{R}\mathsf{R}^\dagger$. This means that for *any* covariance matrix K, there is a matrix R such that the random vector $\boldsymbol{Z} = \mathsf{R}\boldsymbol{W}$ has covariance K and pseudo-covariance 0. This is summed up in the following theorem.

Theorem 2. *A necessary and sufficient condition for a random vector to be a circularly-symmetric jointly-Gaussian random vector is that it has the form $\boldsymbol{Z} = A\boldsymbol{W}$ where $\boldsymbol{W}$ is iid complex Gaussian and A is an arbitrary complex matrix.*

We now have three equivalent characterizations for circularly-symmetric Gaussian random vectors, first, the definition in terms of phase invariance, second, in terms of zero pseudo-covariance, and third, in terms of linear transformations of iid Gaussian vectors. One advantage of the third characterization is that the jointly-Gaussian requirement is automatically met, whereas the other two depend on that as a separate requirement. Another advantage of the third characterization is that the usual motivation for modeling random vectors as circularly symmetric is that they are linear transformations of essentially iid complex Gaussian random vectors.

3 The real and imaginary parts of circularly symmetric random vectors

Since the probability density of a complex random variable or vector is defined in terms of the real and imaginary parts of that variable or vector, we now pause to discuss these relationships. The major reason for using complex vector spaces and complex random vectors

[2]A complex matrix with orthonormal columns is called a unitary matrix.

is to avoid all the detail of the real and imaginary parts, but our intuition comes from $\mathbb{R}^2$ and $\mathbb{R}^3$, and the major source of confusion in treating complex random vectors comes from assuming that $\mathbb{C}^n$ is roughly the same as $\mathbb{R}^n$. This assumption causes additional confusion when dealing with circular symmetry.

Let $\boldsymbol{Z}$ be a random complex n-vector $\boldsymbol{Z} = (Z_1, \dots, Z_n)^\mathsf{T}$ and let the corresponding real random $2n$-vector $\boldsymbol{V}$ consist of the real and imaginary components of $\boldsymbol{Z}$, taken in the order $\boldsymbol{V} = (\Re(Z_1), \dots, \Re(Z_n), \Im(Z_1), \dots, \Im(Z_n))^\mathsf{T}$.

We start by relating the $2n$ by $2n$ real covariance matrix $\mathsf{K}_{\boldsymbol{V}} = \mathsf{E}[\boldsymbol{V}\boldsymbol{V}^\mathsf{T}]$ to the n by n matrix $\mathsf{K}_{\boldsymbol{Z}} = \mathsf{E}[\boldsymbol{Z}\boldsymbol{Z}^\dagger]$. The (real) covariance matrix $\mathsf{K}_{\boldsymbol{V}}$ can then be expressed in block form in terms of the (complex) covariance matrix $\mathsf{K}_{\boldsymbol{Z}}$.

$$\mathsf{K}_{\boldsymbol{V}} = \begin{bmatrix} \mathsf{E}[\Re(\boldsymbol{Z})\Re(\boldsymbol{Z}^\mathsf{T}) & \mathsf{E}[\Re(\boldsymbol{Z})\Im(\boldsymbol{Z}^\mathsf{T}) \\ \\ \mathsf{E}[\Im(\boldsymbol{Z})\Re(\boldsymbol{Z}^\mathsf{T}) & \mathsf{E}[\Im(\boldsymbol{Z})\Im(\boldsymbol{Z}^\mathsf{T}) \end{bmatrix}$$

Assume in what follows that $\boldsymbol{Z}$ is circularly symmetric. The pseudo-covariance matrix, $\mathsf{M}_{\boldsymbol{Z}}$ is then zero, so expressing (2), in matrix form, we get $\mathsf{E}[\Re(\boldsymbol{Z})\Re(\boldsymbol{Z}^\mathsf{T})] = \frac{1}{2}\Re(\mathsf{K}_{\boldsymbol{Z}})$. Using (3), (4), and (5) in the same way, $\mathsf{K}_{\boldsymbol{V}}$ can be expressed as

$$\mathsf{K}_{\boldsymbol{V}} = \begin{bmatrix} \frac{1}{2}\Re(\mathsf{K}_{\boldsymbol{Z}}) & -\frac{1}{2}\Im(\mathsf{K}_{\boldsymbol{Z}}) \\ \\ \frac{1}{2}\Im(\mathsf{K}_{\boldsymbol{Z}}) & \frac{1}{2}\Re(\mathsf{K}_{\boldsymbol{Z}}) \end{bmatrix} \tag{7}$$

It can then be verified by matrix multiplication[3] with (7) that if $\mathsf{K}_{\boldsymbol{Z}}$ is non-singular, then $\mathsf{K}_{\boldsymbol{V}}^{-1}$ is also non-singular and its inverse is given in block form by

$$\mathsf{K}_{\boldsymbol{V}}^{-1} = \begin{bmatrix} 2\Re(\mathsf{K}_{\boldsymbol{Z}}^{-1}) & -2\Im(\mathsf{K}_{\boldsymbol{Z}}^{-1}) \\ \\ 2\Im(\mathsf{K}_{\boldsymbol{Z}}^{-1}) & 2\Re(\mathsf{K}_{\boldsymbol{Z}}^{-1}) \end{bmatrix} \tag{8}$$

We next relate the eigenvalue, eigenvector pairs of $\mathsf{K}_{\boldsymbol{Z}}$ to those of $\mathsf{K}_{\boldsymbol{V}}$. Let λ_j be an eigenvalue of $\mathsf{K}_{\boldsymbol{Z}}$ and let $\boldsymbol{q}_j = (q_{1j}, \dots, q_{nj})^\mathsf{T}$ be a corresponding eigenvector, chosen so that the set of eigenvectors is orthonormal. Let $\boldsymbol{r}_j = (\Re(q_{1j}), \dots \Re(q_{nj}), \Im(q_{1j}), \dots, \Im(q_{nj}))^\mathsf{T}$ be the corresponding eigenvector expressed as a real $2n$-vector.

We now show that $\boldsymbol{r}_j$ is an eigenvector of $\mathsf{K}_{\boldsymbol{V}}$ with the eigenvalue $\lambda_j/2$. Using (7) (which

[3]In more detail, $\mathsf{I}_n = \mathsf{K}_{\boldsymbol{Z}}\mathsf{K}_{\boldsymbol{Z}}^{-1}$ implies that $\Re(\mathsf{K}_{\boldsymbol{Z}})\Re(\mathsf{K}_{\boldsymbol{Z}}^{-1}) - \Im(\mathsf{K}_{\boldsymbol{Z}})\Im(\mathsf{K}_{\boldsymbol{Z}}^{-1}) = \mathsf{I}_n$ and $\Re(\mathsf{K}_{\boldsymbol{Z}})\Im(\mathsf{K}_{\boldsymbol{Z}}^{-1}) + \Im(\mathsf{K}_{\boldsymbol{Z}})\Re(\mathsf{K}_{\boldsymbol{Z}}^{-1}) = 0$. With this, it is seen that the product of (6) and (7) is I_{2n}

depends on the circular symmetry of $\boldsymbol{Z}$),

$$\mathsf{K}_{\boldsymbol{V}}\boldsymbol{r}_j = \begin{bmatrix} \frac{1}{2}\Re(\mathsf{K}_{\boldsymbol{Z}}) & -\frac{1}{2}\Im(\mathsf{K}_{\boldsymbol{Z}}) \\ \frac{1}{2}\Im(\mathsf{K}_{\boldsymbol{Z}}) & \frac{1}{2}\Re(\mathsf{K}_{\boldsymbol{Z}}) \end{bmatrix} \begin{bmatrix} \Re(\boldsymbol{q}_j) \\ \Im(\boldsymbol{q}_j) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\Re(\mathsf{K}_{\boldsymbol{Z}})\Re(\boldsymbol{q}_j) - \frac{1}{2}\Im(\mathsf{K}_{\boldsymbol{Z}})\Im(\boldsymbol{q}_j) \\ \frac{1}{2}\Im(\mathsf{K}_{\boldsymbol{Z}})\Re(\boldsymbol{q}_j) + \frac{1}{2}\Re(\mathsf{K}_{\boldsymbol{Z}})\Im(\boldsymbol{q}_j) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\Re(\mathsf{K}_{\boldsymbol{Z}}\boldsymbol{q}_j) \\ \frac{1}{2}\Im(\mathsf{K}_{\boldsymbol{Z}}\boldsymbol{q}_j) \end{bmatrix} = \begin{bmatrix} \frac{\lambda_j}{2}\Re(\boldsymbol{q}_j) \\ \frac{\lambda_j}{2}\Im(\boldsymbol{q}_j) \end{bmatrix} = \frac{\lambda_j}{2}\boldsymbol{r}_j \qquad (9)$$

In the penultimate step, we used the facts that λ_j is real and that $\boldsymbol{q}_j$ is an eigenvector of $\mathsf{K}_{\boldsymbol{Z}}$.

This specifies n of the eigenvalue/eigenvector pairs of $\mathsf{K}_{\boldsymbol{V}}$, but what about the rest? We next show that the eigenvalues of $\mathsf{K}_{\boldsymbol{V}}$ come in pairs. For each eigenvalue λ_j of $\mathsf{K}_{\boldsymbol{Z}}$, there are two eigenvalues of $\mathsf{K}_{\boldsymbol{V}}$, each equal to $\lambda_j/2$. The corresponding eigenvectors can be chosen as $\boldsymbol{r}_j$ and $\boldsymbol{m}_j$ where $\boldsymbol{m}_j = (-\Im(q_{1j}), \dots, -\Im(\boldsymbol{q}_{nj}), \Re(\boldsymbol{q}_{1j}), \dots, \Re(\boldsymbol{q}_{nj})^{\mathsf{T}}$. It can be seen that $\boldsymbol{m}_j$ is the representation of $i\boldsymbol{q}_j$ as a real $2n$-vector. It can also be verified directly that $\boldsymbol{r}_j$ and $\boldsymbol{m}_j$ (as vectors in $\mathbb{R}^{2n}$) are orthonormal.[4] Next, the calculation in (9), with $\boldsymbol{r}_j$ replaced by $\boldsymbol{m}_j$, shows that $\boldsymbol{m}_j$ is an eigenvector of $\mathsf{K}_{\boldsymbol{V}}$ with eigenvalue $\lambda_j/2$. Finally, let $\boldsymbol{q}_j$ and $\boldsymbol{q}_k$ be orthonormal eigenvectors of $\mathsf{K}_{\boldsymbol{Z}}$. Then the corresponding real $2n$-vectors $\boldsymbol{r}_j$ and $\boldsymbol{r}_k$ are orthonormal, as shown by

$$\boldsymbol{r}_j^{\mathsf{T}}\boldsymbol{r}_k = \Re(\boldsymbol{q}_j^{\mathsf{T}})\Re(\boldsymbol{q}_k) + \Im(\boldsymbol{q}_j^{\mathsf{T}})\Im(\boldsymbol{q}_k) = \Re(\boldsymbol{q}_j^{\dagger}\boldsymbol{q}_k) = \delta_{j,k}$$

Similarly, $(\boldsymbol{r}_j, \boldsymbol{m}_k)$ are orthonormal and $(\boldsymbol{m}_j, \boldsymbol{m}_k)$ are orthonormal. Thus the set $(\boldsymbol{r}_1, \dots, \boldsymbol{r}_n, \boldsymbol{m}_1, \dots, \boldsymbol{m}_n)$ is an orthonormal set and spans $\mathbb{R}^{2n}$.

4 The probability density of circularly-symmetric Gaussian vectors

Assuming that $\mathsf{K}_{\boldsymbol{Z}}$ is positive definite, all its eignevalues $\lambda_1, \dots, \lambda_n$ are positive, so all the eigenvalues of $\mathsf{K}_{\boldsymbol{V}}$ are also positive and $\mathsf{K}_{\boldsymbol{V}}$ is positive definite. The probability density of $\boldsymbol{Z}$ is defined as the joint probability density of its real and imaginary parts, so it is the probability density associated with $\boldsymbol{V}$. Since this is jointly Gaussian and non-singular,

$$f_{\boldsymbol{V}}(\boldsymbol{v}) = \frac{1}{(2\pi)^n\sqrt{\det(\mathsf{K}_{\boldsymbol{V}})}} \exp\frac{-\boldsymbol{v}^{\mathsf{T}}\mathsf{K}_{\boldsymbol{V}}^{-1}\boldsymbol{v}}{2} \qquad (10)$$

[4]One could equally well replace $\boldsymbol{r}_j$ with $e^{i\phi}\boldsymbol{r}_j$ for any real ϕ and replace $\boldsymbol{m}_j$ with $e^{i\phi}\boldsymbol{m}_j$.

This can be expressed directly in terms of the complex random n-vector $\boldsymbol{Z}$. For any sample value $\boldsymbol{z}$ of $\boldsymbol{Z}$, the corresponding real $2n$-vector is $\boldsymbol{v}^\mathsf{T} = (\Re(\boldsymbol{z}^\mathsf{T}), \Im(\boldsymbol{z}^\mathsf{T}))$. Using this and the expression in (8) for $\mathsf{K}_{\boldsymbol{V}}^{-1}$,

$$\boldsymbol{v}^\mathsf{T}\mathsf{K}_{\boldsymbol{V}}^{-1}\boldsymbol{v} = 2\boldsymbol{z}^\dagger\mathsf{K}_{\boldsymbol{Z}}^{-1}\boldsymbol{z} \tag{11}$$

Establishing this equality also uses the fact that $\boldsymbol{z}^\dagger\mathsf{K}_{\boldsymbol{Z}}^{-1}\boldsymbol{z}$ is real. We now recall that each eigenvalue λ_j of $\mathsf{K}_{\boldsymbol{Z}}$ corresponds to two eigenvalues, both $\lambda_j/2$, of $\mathsf{K}_{\boldsymbol{V}}$. Thus

$$\det(\mathsf{K}_{\boldsymbol{V}}) = 2^{2n}(\det \mathsf{K}_{\boldsymbol{Z}})^2 \tag{12}$$

Substituting these relations into (10), we get an expression for $f_{\boldsymbol{Z}}$ in terms of $\mathsf{K}_{\boldsymbol{Z}}$.

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = \frac{1}{\pi^n \det(\mathsf{K}_{\boldsymbol{Z}})} \exp(-\boldsymbol{z}^\dagger\mathsf{K}_{\boldsymbol{Z}}^{-1}\boldsymbol{z}) \tag{13}$$

Note that (10) is valid for any jointly-Gaussian $2n$-vector, whereas (11) and (12), and thus (13), depend on circular symmetry. Next, suppose an arbitrary random vector $\boldsymbol{Z}$ has the density in (13). We have seen that there is a circularly-symmetric Gaussian random vector with the covariance matrix $\mathsf{K}_{\boldsymbol{Z}}$, and this also has the density in (13). Since the density fully describes whether $\boldsymbol{Z}$ is circularly-symmetric jointly Gaussian, we conclude that (13) is a necessary and sufficient condition for a non-singular complex random vector to be circularly-symmetric jointly Gaussian. The following theorem summarizes this.

Theorem 3. *Assume that $\boldsymbol{Z}$ is a complex random n-vector with an arbitrary non-singular covariance $\mathsf{K}_{\boldsymbol{Z}}$. Then (13) is a necessary and sufficient condition for $\boldsymbol{Z}$ to be $\mathcal{CN}(0, \mathsf{K}_{\boldsymbol{Z}})$.*

If $\mathsf{K}_{\boldsymbol{Z}}$ is singular, then one or more components of $\boldsymbol{Z}$ are deterministic linear combinations of the other components, so the most convenient way of specifying $\boldsymbol{Z}$ is by first removing components of $\boldsymbol{Z}$ that are deterministic linear combinations of the remaining components and using (13) for the remaining components.

As with real jointly-Gaussian random vectors, it is often more insightful to express the probability density in terms of the eigenvalues and eigenfunctions of $\mathsf{K}_{\boldsymbol{Z}}$. Letting $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\mathsf{K}_{\boldsymbol{Z}}$ (repeated as necessary) and $\boldsymbol{q}_1, \dots, \boldsymbol{q}_n$ be the corresponding eigenfunctions, we can use (6) to express $\mathsf{K}_{\boldsymbol{Z}}$ as

$$\mathsf{K}_{\boldsymbol{Z}} = \sum_{j=1}^{n} \lambda_j \boldsymbol{q}_j \boldsymbol{q}_j^\dagger$$

Substituting this into (13,

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = \frac{1}{\pi^n \det(\mathsf{K}_{\boldsymbol{Z}})} \exp\left(-\sum_j \boldsymbol{z}^\dagger \boldsymbol{q}_j \lambda_j^{-1} \boldsymbol{q}_j^\dagger \boldsymbol{z}\right)$$

Expressing $\boldsymbol{q}_j^\dagger\boldsymbol{z}$ as the projection of $\boldsymbol{z}$ on $\boldsymbol{q}_j$, *i.e.*, as $\langle \boldsymbol{z}, \boldsymbol{q}_j\rangle$, and recalling that $\det \mathsf{K}_{\boldsymbol{Z}} = \prod_j \lambda_j$, this becomes

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = \prod_{j=1}^{n} \frac{1}{\pi\lambda_j} \exp\left(-|\langle \boldsymbol{z}, \boldsymbol{q}_j\rangle|^2 \lambda_j^{-1}\right) \tag{14}$$

This is the density of n independent circularly-symmetric Gaussian random variables, $(\langle \boldsymbol{Z}, \boldsymbol{q}_1\rangle, \dots, \langle \boldsymbol{Z}, \boldsymbol{q}_n\rangle)$ with variances $\lambda_1, \dots, \lambda_n$ respectively. In other words, expressing $\boldsymbol{Z}$ in the orthonormal basis $\{\boldsymbol{q}_1, \dots, \boldsymbol{q}_n\}$, the variables in this basis are independent circularly-symmetric Gaussian random variables with variances $\lambda_1, \dots, \lambda_n$. This is the same as the analogous result for jointly-Gaussian real random vectors which says that there is always an orthonormal basis in which the variables are Gaussian and independent. This analogy forms the simplest way to (sort of) visualize circularly-symmetric Gaussian vectors – they have the same kind of elliptical symmetry as the real case, except that here, each complex random variable is also circularly symmetric. It should be clear that (14) is also an if-and-only-if condition for circularly-symmetric jointly-Gaussian random vectors.

5 Linear functionals of circularly-symmetric random vectors

Let $\boldsymbol{Z} \sim \mathcal{CN}(0, \mathsf{K}_{\boldsymbol{Z}})$. If some other random vector $\boldsymbol{Y}$ can be expressed as $\boldsymbol{Y} = \mathsf{B}\boldsymbol{Z}$, then $\boldsymbol{Y}$ is also a circularly-symmetric jointly-Gaussian random vector. To see this, represent $\boldsymbol{Z}$ as $\boldsymbol{Z} = \mathsf{A}\boldsymbol{W}$ where $\boldsymbol{W} \sim \mathcal{CN}(0, I)$. Then $\boldsymbol{Y} = \mathsf{BA}\boldsymbol{W}$, so $\boldsymbol{Y} \sim \mathcal{CN}(0, \mathsf{B}\mathsf{K}_{\boldsymbol{Z}}\mathsf{B}^{\dagger})$. This helps show why circular symmetry is important – it is invariant to linear transformations.

If B is 1 by n (*i.e.*, if it is a row vector $\boldsymbol{b}^{\mathsf{T}}$) then $Y = \boldsymbol{b}^{\mathsf{T}}\boldsymbol{Z}$ is a complex rv. Such rv's are called *linear functionals* of $\boldsymbol{Z}$. Thus all linear functionals of a circularly-symmetric jointly Gaussian random vector are circularly-symmetric Gaussian rv's.

Conversely, we now want to show that if all linear functionals of a complex random vector $\boldsymbol{Z}$ are circularly-symmetric Gaussian, then $\boldsymbol{Z}$ must also be circularly-symmetric and jointly-Gaussian. The question of being jointly Gaussian can be separated from that of being circularly symmetric. Thus assume that for all complex n-vectors $\boldsymbol{b}^{\mathsf{T}}$, the complex rv $\boldsymbol{b}^{\mathsf{T}}\boldsymbol{Z}$ is complex Gaussian. Looking at $\boldsymbol{V}$ and $\boldsymbol{b}^{\mathsf{T}}$ as real $2n$ vectors, $\boldsymbol{V} = (\Re(Z_1), \dots, \Re(Z_n), \Im(Z_1), \dots, \Im(Z_n)^{\mathsf{T}}$ and $\boldsymbol{r}^{\mathsf{T}} = (\Re(b_1), \dots, \Re(b_n), \Im(b_1) \dots, \Im(b_n))^{\mathsf{T}}$, we see that $\boldsymbol{r}^{\mathsf{T}}\boldsymbol{V}$ is a Gaussian random variable for all $\boldsymbol{b}$. Since $\boldsymbol{r}$ can be arbitrarily chosen by choosing the real and imaginary components of $\boldsymbol{b}$, we see that all real linear functionals of $\boldsymbol{V}$ are Gaussian. It is known for real random vectors (see for example Section 7.3.6 of PDC08) that $\boldsymbol{V}$ is jointly Gaussian if all its linear functionals are Gaussian random variables. Thus $\boldsymbol{Z}$ is complex jointly Gaussian if all its linear functionals are complex Gaussian random variables.

We could now show that $\boldsymbol{Z}$ is also circularly symmetric and jointly Gaussian if $\boldsymbol{b}^{\mathsf{T}}\boldsymbol{Z}$ is circularly-symmetric Gaussian for all $\boldsymbol{b}$, but it is just as easy, and yields a slightly stronger result, to show that $\boldsymbol{Z} \sim \mathcal{CN}(0, \mathsf{K}_{\boldsymbol{Z}})$ if $\boldsymbol{Z}$ is jointly Gaussian and, in addition, the limited set of linear functionals $Z_j + Z_k$ is circularly symmetric Gaussian for all j, k. If $Z_j + Z_j$ is circularly symmetric for all j, then $\mathsf{E}[Z_j^2] = 0$, so that the main diagonal of $\mathsf{M}_{\boldsymbol{Z}}$ is zero. If in addition, $Z_j + Z_k$ is circularly symmetric, then $\mathsf{E}[(Z_j + Z_k)^2] = 0$. But since $\mathsf{E}[Z_j^2] = \mathsf{E}[Z_k^2] = 0$, $2\mathsf{E}[Z_jZ_k] = 0$. Thus the j, k element of $\mathsf{M}_{\boldsymbol{Z}} = 0$. Thus if $Z_j + Z_k$ is circularly symmetric for all j, k, it follows that $\mathsf{M}_{\boldsymbol{Z}} = 0$ and $\boldsymbol{Z}$ is circularly symmetric.[5]

6 Summary

In summary, we have proved the following theorem.

Theorem 4. *A complex random vector* $\boldsymbol{Z}$ *is* $\mathcal{CN}(0, \mathsf{K}_{\boldsymbol{Z}})$ *if and only if any one of the following conditions is satisfied.*

- $\boldsymbol{Z}$ *is jointly Gaussian and has the same distribution as* $\epsilon^{i\phi}\boldsymbol{Z}$ *for all real* ϕ.
- $\boldsymbol{Z}$ *is zero-mean jointly Gaussian and the pseudo-covariance matrix* $\mathsf{M}_{\boldsymbol{Z}}$ *is zero.*
- $\boldsymbol{Z}$ *can be expressed as* $\boldsymbol{Z} = \mathsf{A}\boldsymbol{W}$ *where* $\boldsymbol{W}$ *is a vector of statistically independent components, all* $\mathcal{CN}(0, 1)$.
- *For non-singular* $\mathsf{K}_{\boldsymbol{Z}}$*, the probability density of* $\boldsymbol{Z}$ *is given in (13). For singular* $\mathsf{K}_{\boldsymbol{Z}}$*, (13) gives the density of* $\boldsymbol{Z}$ *after removal of the deterministically dependent components.*
- *For non-singular* $\mathsf{K}_{\boldsymbol{Z}}$*, the probability density of* $\boldsymbol{Z}$ *is given in (14). For singular* $\mathsf{K}_{\boldsymbol{Z}}$*, (14) gives the density of* $\boldsymbol{Z}$ *after removal of the deterministically dependent components.*
- *All linear functionals of* $\boldsymbol{Z}$ *are complex Gaussian and* $Z_j + Z_k$ *is circularly symmetric for all* j, k.

Note that either all or none of these conditions are satisfied. The significance of the theorem is that any one of the conditions may be used to either establish the circularly-symmetric jointly-Gaussian property or to show that it does not hold.

[5] Example 2 illustrates the strange behavior possible when *vecZ* is circularly symmetric and individually Gaussian but not jointly Gaussian.