

# The Hilbert Transform

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## 1 Definition

The Hilbert transform  $\mathcal{H}[g(t)]$  of a signal  $g(t)$  is defined as

$$\mathcal{H}[g(t)] = g(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\tau)}{t - \tau} d\tau = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(t - \tau)}{\tau} d\tau. \quad (1)$$

The Hilbert transform of  $g(t)$  is the convolution of  $g(t)$  with the signal  $1/\pi t$ . It is the response to  $g(t)$  of a linear time-invariant filter (called a Hilbert transformer) having impulse response  $1/\pi t$ . The Hilbert transform  $\mathcal{H}[g(t)]$  is often denoted as  $\hat{g}(t)$  or as  $[g(t)]^\wedge$ .

A technicality arises immediately. The alert reader will already be concerned with the definition (1) as the integral is improper: the integrand has a singularity and the limits of integration are infinite. In fact, the Hilbert transform is properly defined as the *Cauchy principal value* of the integral in (1), whenever this value exists. The Cauchy principal value is defined—for the first integral in (1)—as

$$\mathcal{H}[g(t)] = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+} \left( \int_{t-1/\epsilon}^{t-\epsilon} \frac{g(\tau)}{t - \tau} d\tau + \int_{t+\epsilon}^{t+1/\epsilon} \frac{g(\tau)}{t - \tau} d\tau \right). \quad (2)$$

We see that the Cauchy principal value is obtained by considering a finite range of integration that is symmetric about the point of singularity, but which excludes a symmetric subinterval, taking the limit of the integral as the length of the interval approaches  $\infty$  while, simultaneously, the length of the excluded interval approaches zero. Henceforth, whenever we write an integral as in (1), we will mean the Cauchy principal value of that integral (when it exists).

## 2 Some Basic Properties

Some obvious properties of the Hilbert transform follow directly from the definition. Clearly the Hilbert transform of a time-domain signal  $g(t)$  is another time-domain signal  $\hat{g}(t)$ . If  $g(t)$  is real-valued, then so is  $\hat{g}(t)$ .

**Linearity:** The Hilbert transform is linear, i.e., if  $a_1$  and  $a_2$  are arbitrary (complex) scalars, and  $g_1(t)$  and  $g_2(t)$  are signals, then

$$[a_1g_1(t) + a_2g_2(t)]^\wedge = a_1\hat{g}_1(t) + a_2\hat{g}_2(t).$$

(This follows immediately from the fact that the Hilbert transform is the output of a linear system.)

**The Hilbert transform of a constant signal:** Note that, for any constant  $c$ , the Hilbert transform of the constant signal  $g(t) = c$  is  $\hat{g}(t) = \hat{c} = 0$ . (See Exercise 2.) From linearity it follows that  $\mathcal{H}[g(t) + c] = \hat{g}(t) + \hat{c} = \hat{g}(t)$ . Thus, like an ideal differentiator, a Hilbert transformer “loses” dc offsets. Later we will define an inverse Hilbert transform which can recover the original signal up to an additive constant (in the same way that integration can undo differentiation only up to an additive constant).

**Time-shifting and time-dilation:** If  $g(t)$  has Hilbert transform  $\hat{g}(t)$ , then  $g(t - t_0)$  has Hilbert transform  $\hat{g}(t - t_0)$ , and  $g(at)$  has Hilbert transform  $\text{sgn}(a)\hat{g}(at)$  (assuming  $a \neq 0$ ).

**Convolution:** The Hilbert transform behaves nicely with respect to convolution, since

$$[g_1(t) * g_2(t)]^\wedge = \hat{g}_1(t) * g_2(t) = g_1(t) * \hat{g}_2(t).$$

To see this, observe from the associative and commutative properties of convolution that  $[g_1(t) * g_2(t)] * \frac{1}{\pi t}$  can be written as  $[g_1(t) * \frac{1}{\pi t}] * g_2(t)$  or as  $g_1(t) * [g_2(t) * \frac{1}{\pi t}]$ .

**Time-derivative:** The Hilbert transform of the derivative of a signal is the derivative of the Hilbert transform, i.e.,

$$\mathcal{H}\left[\frac{d}{dt}g(t)\right] = \frac{d}{dt}\mathcal{H}[g(t)].$$

To see this, recall Leibniz’s Integral Rule, which states that

$$\frac{d}{dc} \int_{a(c)}^{b(c)} f(x, c) dx = \int_{a(c)}^{b(c)} \frac{\partial}{\partial c} f(x, c) dx + f(b, c) \frac{d}{dc} b(c) - f(a, c) \frac{d}{dc} a(c).$$

In particular, if  $a$  and  $b$  are definite limits (independent of  $c$ ), we have

$$\frac{d}{dc} \int_a^b f(x, c) dx = \int_a^b \frac{\partial}{\partial c} f(x, c) dx.$$

Now

$$\begin{aligned} \frac{d}{dt} \mathcal{H}[g(t)] &= \frac{1}{\pi} \frac{d}{dt} \int_{-\infty}^{\infty} \frac{g(t - \tau)}{\tau} d\tau \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g'(t - \tau)}{\tau} d\tau \\ &= \mathcal{H}[g'(t)], \end{aligned}$$

where  $g'(t) = \frac{d}{dt}g(t)$ .

### 3 Interaction with the Fourier Transform

The signal  $1/(\pi t)$  has Fourier transform

$$-j \operatorname{sgn}(f) = \begin{cases} -j, & \text{if } f > 0 \\ 0, & \text{if } f = 0 \\ j, & \text{if } f < 0 \end{cases}$$

If  $g(t)$  has Fourier transform  $G(f)$ , then, from the convolution property of the Fourier transform, it follows that  $\hat{g}(t)$  has Fourier transform

$$\hat{G}(f) = -j \operatorname{sgn}(f)G(f).$$

Thus, the Hilbert transform is easier to understand in the frequency domain than in the time domain: the Hilbert transform does not change the *magnitude* of  $G(f)$ , it changes only the *phase*. Fourier transform values at positive frequencies are multiplied by  $-j$  (corresponding to a phase change of  $-\pi/2$ ) while Fourier transform values at negative frequencies are multiplied by  $j$  (corresponding to a phase change of  $\pi/2$ ). Stated yet another way, suppose that  $G(f) = a + bj$  for some  $f$ . Then  $\hat{G}(f) = b - aj$  if  $f > 0$  and  $\hat{G}(f) = -b + aj$  if  $f < 0$ . Thus the Hilbert transform essentially acts to exchange the real and imaginary parts of  $G(f)$  (while changing the sign of one of them).

**Energy Spectral Density:** Suppose that  $g(t)$  is an energy signal. Then, since  $|\hat{G}(f)| = |G(f)|$ , both  $\hat{G}(f)$  and  $G(f)$  have exactly the same energy spectral density. Thus, for example, if  $G(f)$  is bandlimited to  $B$  Hz then so is  $\hat{G}(f)$ . It also follows that  $\hat{g}(t)$  has exactly the same energy as  $g(t)$ .

**Symmetry Properties:** If  $g(t)$  is real-valued, then  $G(f)$  exhibits Hermitian symmetry, i.e.,  $G(-f) = G^*(f)$ . Of course then  $\hat{G}(-f) = -j \operatorname{sgn}(-f)G(-f) = [-j \operatorname{sgn}(f)G(f)]^* = \hat{G}(f)^*$ , so  $\hat{G}(f)$  also exhibits Hermitian symmetry, as expected.

Let  $g(t)$  be a real-valued signal. Recall that if  $g(t)$  is even (so that  $g(-t) = g(t)$ ) then  $G(f)$  is purely real-valued while if  $g(t)$  is odd (so that  $g(-t) = -g(t)$ ) then  $G(f)$  is purely imaginary-valued. Now if  $G(f)$  is purely real-valued then certainly  $\hat{G}(f)$  is purely imaginary-valued (and vice-versa). Thus if  $g(t)$  is even, then  $\hat{g}(t)$  is odd and if  $g(t)$  is odd, then  $\hat{g}(t)$  is even.

**Orthogonality:** If  $g(t)$  is a real-valued energy signal, then  $g(t)$  and  $\hat{g}(t)$  are orthogonal. To see this recall that

$$\begin{aligned} \langle g(t), \hat{g}(t) \rangle &= \int_{-\infty}^{\infty} g(t)\hat{g}^*(t)dt \\ &= \int_{-\infty}^{\infty} G(f)\hat{G}^*(f)df \\ &= \int_{-\infty}^{\infty} G(f)[-j \operatorname{sgn}(f)G(f)]^*df \\ &= \int_{-\infty}^{\infty} j|G(f)|^2 \operatorname{sgn}(f)df \\ &= 0, \end{aligned}$$

where we have used the property that, since  $|G(f)|^2$  is an even function of  $f$ ,  $|G(f)|^2 \operatorname{sgn}(f)$  is an odd function of  $f$  and hence the value of the integral is zero.

**Low-pass High-pass Products:** Let  $g(t)$  be signal whose Fourier transform satisfies  $G(f) = 0$  for  $|f| \geq W$  and let  $h(t)$  be a signal with  $H(f) = 0$  for  $|f| < W$ . Then

$$\mathcal{H}[g(t)h(t)] = g(t)\hat{h}(t),$$

i.e., to compute the Hilbert transform of the product of a low-pass signal with a high-pass signal, only the high-pass signal needs to be transformed.

To see this, let  $s(t) = g(t)h(t)$  so that

$$S(f) = G(f) * H(f) = G(f) * [H(f) \mathbf{u}(f) + H(f) \mathbf{u}(-f)],$$

where  $\mathbf{u}$  denotes the unit step function. It is easy to see that  $G(f) * (H(f) \mathbf{u}(f))$  is zero if  $f < 0$ ; similarly  $G(f) * (H(f) \mathbf{u}(-f))$  is zero if  $f > 0$ . Thus

$$\begin{aligned} \hat{S}(f) &= -j \operatorname{sgn}(f)S(f) \\ &= -j G(f) * (H(f) \mathbf{u}(f)) + j G(f) * (H(f) \mathbf{u}(-f)) \\ &= G(f) * [-j H(f) \mathbf{u}(f) + j H(f) \mathbf{u}(-f)] \\ &= G(f) * [-j \operatorname{sgn}(f)H(f)] \\ &= G(f) * \hat{H}(f). \end{aligned}$$

An important special case of this arises in the case of QAM modulation. Assuming that  $m_I(t)$  and  $m_Q(t)$  are bandlimited to  $W$  Hz, then, if  $f_c > W$ , we have

$$\mathcal{H}[m_I(t) \cos(2\pi f_c t) + m_Q(t) \sin(2\pi f_c t)] = m_I(t) \sin(2\pi f_c t) - m_Q(t) \cos(2\pi f_c t). \quad (3)$$

**Amplitude-modulated Signals:** The Hilbert transform of a general amplitude-modulated signal is given by [1]

$$\begin{aligned} \mathcal{H}[g(t) \cos(2\pi f_c t + \theta)] &= \left[ g(t) * \frac{\cos(2\pi f_c t)}{\pi t} \right] \cos(2\pi f_c t + \theta) \\ &\quad + \left[ g(t) * \frac{\sin(2\pi f_c t)}{\pi t} \right] \sin(2\pi f_c t + \theta). \end{aligned}$$

To see this, write  $g(t)$  as  $g(t) = g_L(t) + g_H(t)$ , where  $g_L(t)$ , the “low-pass component” of  $g(t)$ , is given as

$$g_L(t) = g(t) * 2f_c \operatorname{sinc}(2f_c t) = g(t) * \frac{\sin(2\pi f_c t)}{\pi t}$$

and  $g_H(t)$ , the “high-pass component” of  $g(t)$ , is given as

$$g_H(t) = g(t) - g_L(t) = g(t) * [\delta(t) - 2f_c \operatorname{sinc}(2f_c t)].$$

Clearly  $g_L(t)$  contains all frequency components of  $g(t)$  from dc up to frequency  $f_c$ , while  $g_H(t)$  contains the remaining components at higher frequencies. We have

$$\begin{aligned} \mathcal{H}[g(t) \cos(2\pi f_c t + \theta)] &= \mathcal{H}[g_L(t) \cos(2\pi f_c t + \theta)] + \mathcal{H}[g_H(t) \cos(2\pi f_c t + \theta)] \\ &= g_L(t) \mathcal{H}[\cos(2\pi f_c t + \theta)] + \mathcal{H}[g_H(t)] \cos(2\pi f_c t + \theta) \end{aligned}$$

where the second equality follows from application of the “low-pass high-pass product” result. Clearly  $\mathcal{H}[\cos(2\pi f_c t + \theta)] = \sin(2\pi f_c t + \theta)$ . It remains to find  $\mathcal{H}[g_H(t)]$ . This is given as

$$\begin{aligned} \mathcal{H}[g_H(t)] &= g(t) * [\delta(t) - 2f_c \operatorname{sinc}(2f_c t)] * \frac{1}{\pi t} \\ &= g(t) * \left[ \frac{1}{\pi t} - 2f_c \operatorname{sinc}(2f_c t) * \frac{1}{\pi t} \right] \\ &\stackrel{(a)}{=} g(t) * \left[ \frac{1}{\pi t} - 2f_c \operatorname{sinc}(f_c t) \sin(\pi f_c t) \right] \\ &= g(t) * \left[ \frac{1}{\pi t} (1 - 2 \sin^2(\pi f_c t)) \right] \\ &= g(t) * \frac{\cos(2\pi f_c t)}{\pi t}, \end{aligned}$$

and the main result follows directly. In the equality (a), we have used the fact that  $\mathcal{H}[\operatorname{sinc}(t)] = \operatorname{sinc}(t/2) \sin(\pi t/2)$ ; see Exercise 6.

**Inverse Hilbert Transform:** Note that

$$[\hat{G}(f)]^\wedge = (-j \operatorname{sgn}(f))^2 G(f) = -\operatorname{sgn}^2(f)G(f).$$

Except at  $f = 0$ ,  $\operatorname{sgn}^2(f) = 1$ . Thus, unless  $G(f)$  has some sort of singularity (e.g., a delta function) at  $f = 0$ , we get  $\mathcal{H}[\mathcal{H}[g(t)]] = -g(t)$ . A delta function at  $f = 0$  corresponds to a nonzero dc offset; which, as already remarked upon, is lost by the Hilbert transform. Thus, assuming that  $g(t)$  has zero mean<sup>1</sup>, we may recover  $g(t)$  from  $\hat{g}(t)$ : the inverse Hilbert transform is given by applying the Hilbert transform again, and negating the result:

$$g(t) = -\mathcal{H}[\hat{g}(t)] = -\hat{g}(t) * \frac{1}{\pi t}.$$

In general, we have, for some constant  $c$ ,

$$g(t) = -\hat{g}(t) * \frac{1}{\pi t} + c.$$

Zero-mean signals  $g(t)$  and  $\hat{g}(t)$  are often referred to as a Hilbert transform pair. For every Hilbert transform pair  $g(t)$  and  $\hat{g}(t)$  there is also the *dual* pair  $\hat{g}(t)$  and  $-g(t)$ . Table 1 lists some Hilbert transform pairs.

Fig. 1 plots  $\operatorname{rect}(t)$  and  $\mathcal{H}[\operatorname{rect}(t)] = \frac{1}{\pi} \ln|(2t + 1)/(2t - 1)|$ . (The rectangular pulse  $\operatorname{rect}(t)$  is defined as  $\mathbf{u}(t + 1/2) - \mathbf{u}(t - 1/2)$ , where  $\mathbf{u}(t)$  is the unit step.)

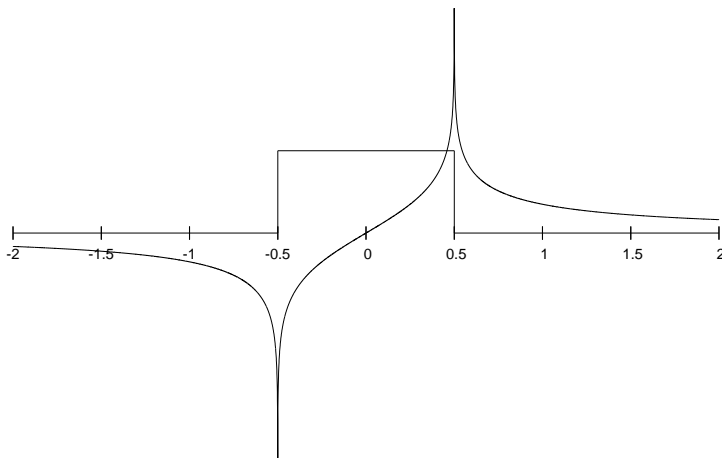


Figure 1: The function  $\operatorname{rect}(t)$  and its Hilbert transform  $\frac{1}{\pi} \ln|(2t + 1)/(2t - 1)|$ .

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<sup>1</sup>The mean of  $g(t)$  is defined as  $\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t) dt$ , when this limit exists.

Table 1: Hilbert transform pairs. (See also [2].)

$g(t)$	$\hat{g}(t)$
$a_1g_1(t) + a_2g_2(t); a_1, a_2 \in \mathbb{C}$	$a_1\hat{g}_1(t) + a_2\hat{g}_2(t)$
$h(t - t_0)$	$\hat{h}(t - t_0)$
$h(at); a \neq 0$	$\text{sgn}(a)\hat{h}(at)$
$\frac{d}{dt}h(t)$	$\frac{d}{dt}\hat{h}(t)$
$\delta(t)$	$\frac{1}{\pi t}$
$e^{jt}$	$-je^{jt}$
$e^{-jt}$	$je^{-jt}$
$\cos(t)$	$\sin(t)$
$\text{rect}(t)$	$\frac{1}{\pi} \ln (2t + 1)/(2t - 1) $
$\text{sinc}(t)$	$\frac{\pi t}{2} \text{sinc}^2(t/2) = \sin(\pi t/2) \text{sinc}(t/2)$
$1/(1 + t^2)$	$t/(1 + t^2)$

## 4 Single-sideband Modulation

For any signal  $g(t)$ , let

$$\begin{aligned} g_+(t) &= \frac{1}{2}[g(t) + j\hat{g}(t)] \\ g_-(t) &= \frac{1}{2}[g(t) - j\hat{g}(t)] \end{aligned}$$

be two complex-valued signals associated with  $g(t)$ . The significance of these two signals can be seen from their Fourier transforms. We have

$$\begin{aligned} G_+(f) &= \frac{1}{2}[G(f) + j\hat{G}(f)] = \frac{1}{2}[G(f) - j^2 \text{sgn}(f)G(f)] = G(f)\frac{1}{2}[1 + \text{sgn}(f)] \\ &= G(f)\mathbf{u}(f) \end{aligned}$$

where  $\mathbf{u}(f)$  is the unit step function, and, similarly,

$$G_-(f) = G(f)\mathbf{u}(-f).$$

Thus  $g_+(t)$  has spectral components (equal to those of  $g(t)$ ) at *positive frequencies only*, i.e.,  $g_+(t)$  has a right-sided spectrum. Similarly,  $g_-(t)$  has spectral components (equal to those of  $g(t)$ ) at negative frequencies only and hence has a left-sided spectrum. These spectra are illustrated in Fig. 2.

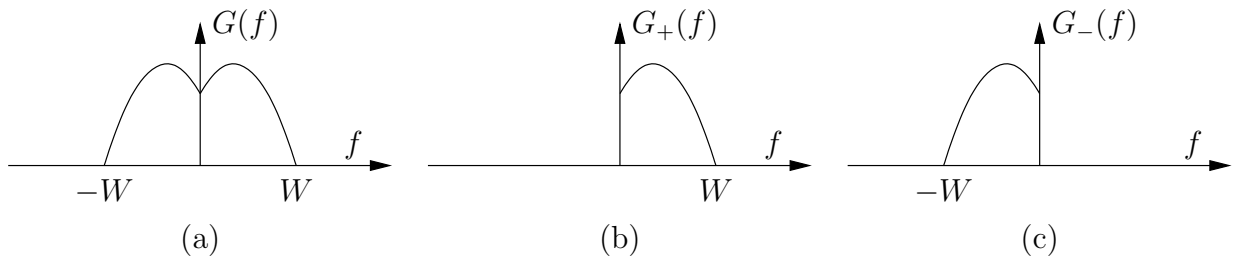


Figure 2: Signal spectra: (a)  $G(f)$ , (b) the right-sided spectrum  $G_+(f)$ , (c) the left-sided spectrum  $G_-(f)$ .

It is now straightforward to express upper- and lower-sideband signals in terms of  $g_+(t)$  and  $g_-(t)$ . Let  $g(t)$  be the modulating signal, assumed bandlimited to  $W$  Hz, and let  $f_c > W$  be the carrier frequency. In the frequency domain, the upper sideband signal is given by

$$S_{\text{USB}}(f) = G_+(f - f_c) + G_-(f + f_c),$$

and the lower sideband signal is given by

$$S_{\text{LSB}}(f) = G_-(f - f_c) + G_+(f + f_c),$$

as sketched in Fig. 3 below.



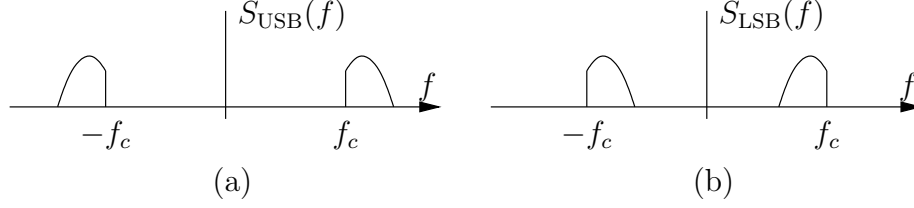


Figure 3: Single-sideband spectra: (a) upper-sideband, (b) lower-sideband.

It follows from the frequency-shifting property of the Fourier transform that

$$\begin{aligned}
 s_{\text{USB}}(f) &= g_+(t) \exp(j2\pi f_c t) + g_-(t) \exp(-j2\pi f_c t) \\
 &= \frac{1}{2}(g(t) + j\hat{g}(t)) \exp(j2\pi f_c t) + \frac{1}{2}(g(t) - j\hat{g}(t)) \exp(-j2\pi f_c t) \\
 &= g(t) \frac{1}{2}[\exp(j2\pi f_c t) + \exp(-j2\pi f_c t)] + \hat{g}(t) \frac{1}{2}[j \exp(j2\pi f_c t) - j \exp(-j2\pi f_c t)] \\
 &= g(t) \cos(2\pi f_c t) - \hat{g}(t) \sin(2\pi f_c t).
 \end{aligned}$$

A similar derivation shows that

$$s_{\text{LSB}}(f) = g(t) \cos(2\pi f_c t) + \hat{g}(t) \sin(2\pi f_c t).$$

Thus we see that single-sideband modulation can be regarded and implemented as a form of quadrature amplitude modulation (QAM), with the modulating signal  $g(t)$  placed in the in-phase channel and the Hilbert transform of  $g(t)$  (or its negative) placed in the quadrature channel. A block diagram illustrating this approach is given in Fig. 4.

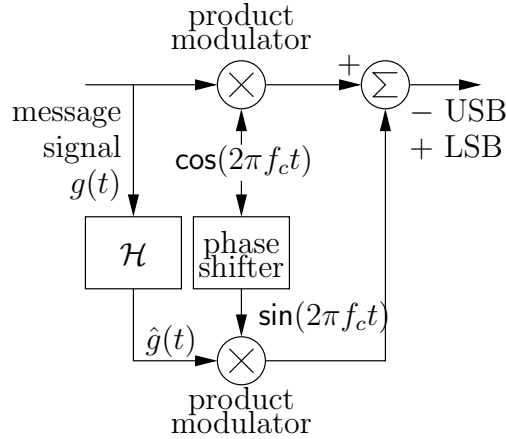


Figure 4: Generation of an SSB-modulated signal.

## 5 Exercises

1. Given an expression for the Cauchy principal value of the second integral in (1).
2. Show that  $\hat{c} = 0$  for any constant  $c$ .
3. Find the Hilbert transform of  $g(t) = 1/t$ .
4. Verify that  $\mathcal{H}[\text{rect}(t)] = \frac{1}{\pi} \ln|(2t+1)/(2t-1)|$ .
5. Verify that  $\mathcal{H}[1/(1+t^2)] = t/(1+t^2)$ .
6. Verify that  $\mathcal{H}[\text{sinc}(t)] = \sin(\pi t/2) \text{sinc}(t/2)$ .
7. If

$$h_L(t) = \frac{\sin(2\pi Wt)}{\pi t}$$

is the impulse response of an ideal lowpass filter with cutoff frequency  $W$ , what operation does the LTI system with impulse response

$$h_H(t) = \frac{\cos(2\pi Wt)}{\pi t}$$

perform? Show that  $h_H(t)$  implements the Hilbert transform of the output of an ideal highpass filter with cutoff frequency  $W$ .

## 6 Solutions to Exercises

- 1.

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(t-\tau)}{\tau} d\tau = \lim_{\epsilon \rightarrow 0^+} \left( \frac{1}{\pi} \int_{-1/\epsilon}^{-\epsilon} \frac{g(t-\tau)}{\tau} d\tau + \frac{1}{\pi} \int_{\epsilon}^{1/\epsilon} \frac{g(t-\tau)}{\tau} d\tau \right).$$

2. We have

$$\begin{aligned} \hat{c} &= \lim_{\epsilon \rightarrow 0^+} \left( \frac{1}{\pi} \int_{-1/\epsilon}^{-\epsilon} \frac{c}{\tau} d\tau + \frac{1}{\pi} \int_{\epsilon}^{1/\epsilon} \frac{c}{\tau} d\tau \right) \\ &= \lim_{\epsilon \rightarrow 0^+} \left( \frac{-1}{\pi} \int_{\epsilon}^{1/\epsilon} \frac{c}{\tau} d\tau + \frac{1}{\pi} \int_{\epsilon}^{1/\epsilon} \frac{c}{\tau} d\tau \right) \\ &= \lim_{\epsilon \rightarrow 0^+} \left( \frac{1}{\pi} \int_{\epsilon}^{1/\epsilon} \left( \frac{c}{\tau} - \frac{c}{\tau} \right) d\tau \right) \\ &= 0 \end{aligned}$$

3. We have  $\mathcal{H}[\delta(t)] = \frac{1}{\pi t}$ , and  $\mathcal{H}[\frac{1}{\pi t}] = -\delta(t)$ . Thus  $\mathcal{H}[1/t] = -\pi\delta(t)$ .

4. Let  $g(t) = \pi \text{rect}(t)$ . Then

$$\hat{g}(t) = \int_{t-1/2}^{t+1/2} \frac{1}{\tau} d\tau.$$

There are four cases.

- In case  $t > 1/2$ , this integral evaluates to  $\ln((t + 1/2)/(t - 1/2))$ .
- By symmetry, in case  $t < -1/2$ , this integral evaluates to  $-\ln((t - 1/2)/(t + 1/2))$ , which can be written as  $\ln((t + 1/2)/(t - 1/2))$ .
- In case  $0 < t < 1/2$ , the region of integration spans the singularity at  $\tau = 0$ . The Cauchy principal value of the integral over the interval  $[t - 1/2, -t + 1/2]$  is zero; therefore what remains is the integral over the interval  $(-t + 1/2, t + 1/2)$ , which evaluates to  $\ln((t + 1/2)/(-t + 1/2))$ .
- By symmetry, in case  $-1/2 < t < 0$ , we find that the integral over the interval  $[-t - 1/2, t + 1/2]$  is zero; therefore what remains is the integral over the interval  $[t - 1/2, -t - 1/2]$ , which evaluates to  $-\ln((-t + 1/2)/(t + 1/2)) = \ln((t + 1/2)/(-t + 1/2))$ .

These four cases can be combined by writing

$$\hat{g}(t) = \ln|(t + 1/2)/(t - 1/2)| = \ln|(2t + 1)/(2t - 1)|.$$

5. Let  $g(t) = 2/(1+t^2)$ , so that  $g(t) = \frac{1}{1-jt} + \frac{1}{1+jt}$ . We have  $\mathcal{F}[1/(1-jt)] = 2\pi e^{-2\pi f} \mathbf{u}(f)$ . By the time-reversal property of the Fourier transform, we get  $\mathcal{F}[1/(1+jt)] = 2\pi e^{2\pi f} \mathbf{u}(-f)$ , and hence

$$G(f) = 2\pi e^{-2\pi f} \mathbf{u}(f) + 2\pi e^{2\pi f} \mathbf{u}(-f).$$

Multiplying by  $-j \text{sgn}(f)$ , we get

$$\hat{G}(f) = -j 2\pi e^{-2\pi f} \mathbf{u}(f) + j 2\pi e^{2\pi f} \mathbf{u}(-f),$$

so

$$\hat{g}(t) = \frac{-j}{1-jt} + \frac{j}{1+jt} = \frac{2t}{1+t^2}.$$

6. From the identity  $\text{sinc}(t) = \text{sinc}(t/2) \cos(\pi t/2)$ , apply the low-pass high-pass product result to get  $\mathcal{H}[\text{sinc}(t)] = \text{sinc}(t/2) \sin(\pi t/2)$ .

7. An ideal highpass filter with cutoff frequency  $W$  has frequency response  $H(f) = 1 - \text{rect}(f/2W)$ , and hence has impulse response  $\delta(t) - 2W \text{sinc}(2Wt)$ . Convolution this

with  $1/(\pi t)$  (to get the Hilbert transform) we get

$$\begin{aligned}h_H(t) &= \frac{1}{\pi t} - 2W \operatorname{sinc}(2Wt) * \frac{1}{\pi t} \\ &= \frac{1}{\pi t} - 2W \operatorname{sinc}(Wt) \sin(\pi Wt) \\ &= \frac{1}{\pi t} (1 - 2 \sin^2(\pi Wt)) \\ &= \frac{\cos(2\pi Wt)}{\pi t}\end{aligned}$$

## References

- [1] S. L. Hahn, "Comments on 'A Tabulation of Hilbert Transforms for Electrical Engineers'," *IEEE Trans. on Commun.*, vol. 44, p. 768, July 1996.
- [2] S. L. Hahn, "Hilbert transforms," in *The Transforms and Applications Handbook* (A. Poularakis, Ed.), Boca Raton FL: CRC Press, 1996, ch. 7.