Rayleigh and Rician Fading

Consider two independent normal random variables $X \sim N(m_1, \sigma^2)$ and $Y \sim N(m_2, \sigma^2)$. Let us define a complex Gaussian random variable Z via : $Z = X + jY$. The envelope and phase of the random variable Z are defined as:

$$R = \sqrt{X^2 + Y^2} \quad , \quad \Theta = \tan^{-1}\left(\frac{Y}{X}\right).$$

The unique solution of these equations for X and Y in terms of R and Θ are given by the polar coordinate representation:

$$X = R\cos\Theta \ , \ Y = R\sin\Theta.$$

The Jacobian for the area transformation is given by:

$$dxdy = rdrd\theta \quad \Longleftrightarrow \quad \left|\mathbf{J}\left(\frac{r,\theta}{x,y}\right)\right| = \frac{1}{r}.$$

Using the Jacobian method for the joint distribution of R and θ we obtain:

$$f_{R,\Theta}(r,\theta) = \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2+s^2}{2\sigma^2}\right)\exp\left(-r\frac{m_1\cos\theta + m_2\sin\theta}{\sigma^2}\right)u(r),\ \theta \in [-\pi,\pi].$$

$I_o(x)$ is the modified zero-th order Bessel function of the first kind defined by

$$I_o(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-x\cos\theta}d\theta.$$

The parameter s is the non-centrality parameter of the faded envelope and is defined as:

$$s = \sqrt{m_1^2 + m_2^2}.$$

The Rice factor K of the faded envelope is a measure of the power in the faded envelope that has been produced by the means of X and Y and is defined as :

$$K = \frac{m_1^2 + m_2^2}{2\sigma^2} = \frac{s^2}{2\sigma^2}.$$

Upon integrating this expression over $\theta \in [-\pi,\pi]$ from $f_{R,\Theta}(r,\theta)$, we obtain the marginal distribution over the envelope:

$$f_R(r) = \frac{r}{\sigma^2}\exp\left(-\frac{r^2+s^2}{2\sigma^2}\right)I_o\left(\frac{rs}{\sigma^2}\right)u(r).$$

In general the marginal distribution for the phase is dependent on θ and is obtained by integrating out R from the joint distribution:

$$f_\Theta(\theta) = \frac{1}{2\pi} \exp\left(-\frac{s^2}{2\sigma^2}\right) + \frac{\tilde{m}}{2\pi\sigma} \exp\left(\frac{\tilde{m}^2 - s^2}{2\sigma^2}\right) Q\left(-\frac{\tilde{m}}{\sigma}\right), \theta \in [-\pi, \pi]$$

where the parameter $\tilde{m}$ is defined as :

$$\tilde{m} = m_1 cos\theta + m_2 sin\theta = \sqrt{m_1^2 + m_2^2} \cos\left(\theta + \cos^{-1}\left(\frac{m_1}{m_2}\right)\right),$$

and $Q(x)$ denotes the normalized Gaussian tail probability function defined via:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy.$$

When $K = 0$, the marginal for the phase is :

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & \theta \in [-\pi, \pi] \\ 0 & \text{otherwise.} \end{cases}$$

In other words the phase of the random variable Z is uniformly distributed, i.e., $\Theta \sim U([-\pi, \pi])$ when the rice factor of the envelope is $K = 0$.

For the specific case when the $K = 0$, the Rician faded envelope reduces down to the Rayleigh faded envelope:

$$f_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) u(r)$$