Proof

To prove the equation above, we need three basic trig identities

$$\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ 2\cos A \cos B &= \cos(A-B) + \cos(A+B) \\ 2\sin A \sin B &= \cos(A-B) - \cos(A+B) \end{aligned}$$

and three Bessel function identities

$$\begin{aligned} \cos(z\sin\theta) &= J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z)\cos(2k\theta) \\ \sin(z\sin\theta) &= 2\sum_{k=0}^{\infty} J_{2k+1}(z)\sin((2k+1)\theta) \\ J_{-n}(z) &= (-1)^n J_n(z) \end{aligned}$$

The Bessel function identities above can be found in Abramowitz and Stegun as equations 9.1.42, 9.1.43, and 9.1.5.

And now the proof. We start with

$$\cos(2\pi f_c t + \beta\sin(2\pi f_m t))$$

and apply the sum identity for cosines to get

$$\cos(2\pi f_c t)\cos(\beta\sin(2\pi f_m t)) - \sin(2\pi f_c t)\sin(\beta\sin(2\pi f_m t))$$

Now let's take the first term

$$\cos(2\pi f_c t)\cos(\beta\sin(2\pi f_m t))$$

and apply one of our Bessel identities to expand it to

$$J_0(\beta)\cos(2\pi f_c t) + \sum_{k=1}^{\infty} J_{2k}(\beta)\left\{\cos(2\pi(f_c - 2kf_m)t) + \cos(2\pi(f_c + 2kf_m)t)\right\}$$

which can be simplified to

$$\sum_{n \text{ even}} J_n(\beta) \cos(2\pi(f_c + n f_m)t)$$

where the sum runs over all even integers, positive and negative.

Now we do the same with the second half of the cosine sum. We expand

$$\sin(2\pi f_c t) \sin(\beta \sin(2\pi f_m t))$$

to

$$\sum_{k=1}^{\infty} J_{2k+1}(\beta) \left\{ \cos(2\pi(f_c - (2k+1) f_m)t) - \cos(2\pi(f_c + (2k+1) f_m)t) \right\}$$

which simplifies to

$$-\sum_{n \,\text{odd}} J_n(\beta) \cos(2\pi(f_c + n f_m)t)$$

where again the sum is over all (odd this time) integers. Combining the two halves gives our result

$$\cos(2\pi f_c t + \beta \sin(2\pi f_m t)) = \sum_{k=-\infty}^{\infty} J_k(\beta) \cos(2\pi(f_c + k f_m)t)$$