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The mathematics behind Jost Bürgi's method for calculating sine tables

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Jost Bürgi (1552–1632) is not only one of the first persons who set up tables of logarithms for practical use but also invented a method to calculate highly precise tables of the sine function with small increment of the argument, and this with only a modest amount of calculations. Bürgi's manuscript on his method, however, was never published by him and got lost for almost four centuries until Menso Folkerts discovered it, cf. [2]. The article [1] on this manuscript gives a proof for the correctness of this method which, however, relies on the Theorem of Perron-Frobenius that is over three centuries younger than Bürgi's method. The present contribution discusses how Bürgi may have come to his method and what considerations may have convinced him that it should work.

1 Jost Bürgi and his "artificium"

Having been born on February 28, 1552 at the litte Swiss town of Lichtensteig (Canton St. Gallen), Jost Bürgi worked as a maker of clocks and scientific instruments, astronomer and mathematician at the court of the landgrave of Hessen-Kassel at Kassel from 1579 to 1604 and at the imperial court at Prague from 1604 to 1631. Around 1587 Bürgi began to make practical use of logarithms but published his tables of logarithms only in 1620. On January 31, 1632 he died at Kassel.

During a stay at Prague in July 1592 Bürgi handed over his manuscript "Fundamentum astronomiae" to Emperor Rudolf II. which contained an algorithm, called "artificium", for the calculation of sine tables. According to reports of contemporaries of Bürgi, the "artificium" enabled him to calculate sine tables of arbitrarily high precision with the argument increasing in steps of 1 angular minute or even 2 angular seconds and this within a modest time for the calculation. The "Fundamentum astronomiae", however, got lost for 4 centuries, was (re)discovered by Menso Folkerts only in 1991/2013, and published by Dieter Launert in 2015 [2].

2 The "artificium" and its proof by means from modern mathematics

In order to calculate the values of the sine function for arguments multiples of $\Delta := 90^\circ/N$ apart in the intervall between 0° and 90° with N a nonzero natural number, set $a_0^{(0)} := 0$ and choose positive numbers $a_1^{(0)}, a_2^{(0)}, \ldots, a_{N-1}^{(0)}, a_N^{(0)}$. Then define inductively numbers $a_j^{(k+1)}$ for $j = 0, \ldots, N$ and $b_j^{(k)}$ for $j = 1, \ldots, N$ by

$$b_N^{(k)} := a_N^{(k)}/2 \qquad \text{and} \qquad b_j^{(k)} := a_j^{(k)} + b_{j+1}^{(k)}, \quad \text{for} \quad j = N-1, N-2, \ldots, 2, 1,$$

and

$$a_0^{(k+1)} := 0 \qquad \text{and} \qquad a_j^{(k+1)} := b_j^{(k)} + a_{j-1}^{(k+1)} \quad \text{for} \quad j = 1, 2, \ldots, N-1, N.$$

Then Bürgi observed at the latest in 1587:

For each $j \in \{0, \ldots, N\}$ the quotient $\left(a_j^{(k)} \;/\; a_N^{(k)}\right)_k$ approximates $\sin(j\Delta)$ arbitrarily well for k large enough.

The term "observed" has been chosen deliberately since Bürgi was not too clear in his statement whether he had a proof of the abovementioned fact [2, p. 58].

At any rate, the statement is correct as Andreas Thom has proven [1, pp. 143–146]: The iteration step from the vector $(a_j^{(k)})_{1\le j\le N}$ to the vector $(a_j^{(k+1)})_{1\le j\le N}$ is given by multiplication with a matrix which has only positive entries and therefore, by the Perron-Frobenius theorem, a dominant eigenvalue. A straightforward calculation gives that the vector $(\sin\Delta, \sin 2\Delta, \ldots, \sin(N-1)\Delta, \sin N\Delta)$ is an eigenvector belonging to this eigenvalue.

The fact that a result from the beginning of the 20th century is used in order to show that an algorithm from the end of the 16th century correctly works has led to evaluations differing from

> "[W]e must admit that Bürgi's insight anticipates some aspects of ideas and developments that came to full light only at the beginning of the 20th century." [1, p. 147]

to

> "It seems unlikely to us that Bürgi had a proof, or that he sought one." [3, p. 5].

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3 The sine function in the "artificium"

Bürgi knew the formula

$$\sin\alpha\cdot\sin\beta=\tfrac{1}{2}\left(\sin\left(90^\circ-\alpha+\beta\right)-\sin\left(90^\circ-\alpha-\beta\right)\right)$$

and praised it as a means to reduce multiplication to addition and subtraction and the use of a sine table ("prosthaphaeresis").

For $j=1,\dots,N-1(,N)$ this formula implies

$$\sin j\Delta=\sin(j-1)\Delta+2\sin\tfrac{\Delta}{2}\cdot\cos\left(j-\tfrac{1}{2}\right)\Delta \qquad \text{and} \qquad \cos\left(j-\tfrac{1}{2}\right)\Delta=\cos\left(j+\tfrac{1}{2}\right)\Delta+2\sin\tfrac{\Delta}{2}\cdot\sin j\Delta.$$

If one sets $a_j:=\sin j\Delta$ in Bürgi's scheme and puts $r:=2\sin\frac{\Delta}{2}$, then a double induction first for j and then for k shows that

$$b_j^{(k)}=r^{-(2k+1)}\left(\cos\left(j+\tfrac{1}{2}\right)\Delta+r\cdot\sin j\Delta\right)=r^{-(2k+1)}\cos\left(j-\tfrac{1}{2}\right)\Delta \quad \text{for} \quad j=1,\dots,,N-1(,N)$$

and

$$a_j^{(k)}=r^{-2k}\left(\sin(j-1)\Delta+r\cdot\cos\left(j-\tfrac{1}{2}\right)\Delta\right)=r^{-2k}\sin j\Delta \quad \text{for} \quad j=(0,)1,\dots,N \quad \text{and all} \quad k.$$

4 The approximation procedure of the "artificium"

Since one can normalize the sine by $\sin 90^\circ=1$, one only needs to know the proportions of the different values of sine and cosine and can therefore ignore the constant factors $r^{-(2k+1)}$ and r^{-2k} on the right hand side of the above equations. Then the step from the $a_j^{(k)}$ to the $b_j^{(k)}$ means that the values of the cosine are calculated from the values of the sine as

$$\cos\left(j-\tfrac{1}{2}\right)\Delta=\cos\left(j+\tfrac{1}{2}\right)\Delta+r\cdot\sin j\Delta$$

for $j=1,\dots,N$, similarly for the step from the $b_j^{(k)}$ to the $a_j^{(k+1)}$.

If one does not start off with the exact values $\sin j\Delta$ as $a_j^{(0)}$ but only with approximations $s(j\Delta)$ of them then the algorithm does not give exact values for the cosine but also only approximations $c\left(\left(j-\frac{1}{2}\right)\Delta\right)$ and the above equation becomes

$$c\left(\left(j-\tfrac{1}{2}\right)\Delta\right)=c\left(\left(j+\tfrac{1}{2}\right)\Delta\right)+R\cdot s(j\Delta)$$

with $R:=2s\left(\frac{\Delta}{2}\right)$. For $N>1$ one has $r=2\sin\frac{\Delta}{2}<1$. Therefore, for any reasonable approximation s of the sine also $R=2s\left(\frac{\Delta}{2}\right)<1$ will hold.

So, if one has an exact value or at least a sufficiently good approximation for one value of the cosine, say, $\cos\left(j+\frac{1}{2}\right)\Delta$, then also the values $c(\alpha)$ for angles α neighboring $\left(j+\frac{1}{2}\right)\Delta$ will differ only by a smaller amount from the correct values of the cosine than the values of s differ from the values of the sine. Necessary for this approximation is, of course, that

- one advances only as far away from the position j that the errors coming from the $s(\alpha)$ do not accumulate too much and
- the value $s(j\Delta)$ is small enough so that the error coming from R is under control.

(The argument for going from the $b_j^{(k)}$ to the $a_j^{(k+1)}$ runs a little bit different since one starts with the exact value $\sin 0^\circ=0$.)

5 Conclusion

And, in fact, Bürgi was confronted with several situations in which exactly these two conditions are fulfilled:

- He had to calculate $\sin\alpha$ if $\sin 3\alpha$ and hence $\cos 3\alpha$ were known, in particular $\sin 1^\circ$ from $\sin 3^\circ$ (where the latter could be calculated by the four basic operations of arithmetic and the extraction of square roots).
- Similar but a little bit more complicated was to calculate $\sin\alpha$ if $\sin 5\alpha$ was known, e. g., $\sin 12'$ from $\sin 1^\circ$.
- Furthermore, Bürgi did not use his "artificium" in order to calculate a sine table with an increment of one angular minute. To this purpose, instead, he used an approximation for the values of the sine function for the first 60 angular minutes deduced from $\sin 1^\circ$.

For these situations both numerical evidence and simple estimates showed that the approximation procedure works. Numerical evidence then may have made Bürgi generalize it to all values between 0° and 90°.

References

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