ElMag Zusammenfassung

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1 Vector Analysis Recap

1.1 Partial derivatives

Definition 1 (Partial derivative)**.** A vector valued function $f : \mathbb{R}^m \to \mathbb{R}$, with $\mathbf{v} \in \mathbb{R}^m$, has a partial derivative with respect to v_i defined as

$$\partial_{v_i} f(\mathbf{v}) = \frac{\partial f}{\partial v_i} = \lim_{h\to 0} \frac{f(\mathbf{v} + h\mathbf{e}_i) - f(\mathbf{v})}{h}$$

Theorem 1 (Integration of partial derivatives)**.** Let $f : \mathbb{R}^m \to \mathbb{R}$ be a partially differentiable function of many x_i. When x_i is *indipendent* with respect to all other x_j $(0 < j \leq m, j \neq i)$ then

$$\int \partial_{x_i} f \, dx_i = f + C,$$

where C is a function of $x_1, \dots, x_m$ but not of x_i.

To illustrate the previous theorem, in a simpler case with $f(x, y)$, we get

$$\int \partial_x f(x, y) \, dx = f(x, y) + C(y).$$

Beware that this is valid only if x and y are indipendent. If there is a relation $x(y)$ or $y(x)$ the above does not hold.

1.2 Vector derivatives

Definition 2 (Gradient vector)**.** The *gradient* of a function $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^m$ is a column vector containing the partial derivatives in each direction.

$$\nabla f(\mathbf{x}) = \sum_{i=1}^{m} \partial_{x_i} f(\mathbf{x}) \mathbf{e}_i = \begin{pmatrix} \partial_{x_1} f(\mathbf{x}) \\ \vdots \\ \partial_{x_m} f(\mathbf{x}) \end{pmatrix}$$

Theorem 2 (Gradient in curvilinear coordinates)**.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field. In cylindrical coordinates (r, ϕ, z)

$$\nabla f = \hat{\mathbf{r}} \, \partial_r f + \hat{\boldsymbol{\phi}} \, \frac{1}{r} \partial_\phi f + \hat{\mathbf{z}} \, \partial_z f,$$

and in spherical coordinates (r, θ, ϕ)

$$\nabla f = \hat{\mathbf{r}} \, \partial_r f + \hat{\boldsymbol{\theta}} \, \frac{1}{r} \partial_\theta f + \hat{\boldsymbol{\phi}} \, \frac{1}{r \sin\theta} \partial_\phi f.$$

Definition 3 (Divergence)**.** Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$ be a vector field. The divergence of $\mathbf{F} = (F_{x_1}, \dots, F_{x_m})^t$ is

$$\nabla \cdot \mathbf{F} = \sum_{i=1}^{m} \partial_{x_i} F_{x_i},$$

as suggested by the (ab)use of the dot product notation.

Theorem 3 (Divergence in curvilinear coordinates)**.** Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a field. In cylindrical coordinates (r, ϕ, z)

$$\nabla \cdot \mathbf{F} = \frac{1}{r} \partial_r (r F_r) + \frac{1}{r} \partial_\phi F_\phi + \partial_z F_z,$$

and in spherical coordinates (r, θ, ϕ)

$$\begin{aligned} \nabla \cdot \mathbf{F} = \frac{1}{r^2} \partial_r (r^2 F_r) + \frac{1}{r \sin\theta} \partial_\theta (\sin\theta F_\theta) \\ + \frac{1}{r \sin\theta} \partial_\phi F_\phi \end{aligned}$$

Theorem 4 (Divergence theorem, Gauss's theorem)**.** Because the flux on the boundary ∂V of a volume V contains information of the field inside of V, it is possible relate the two with

$$\int_V \nabla \cdot \mathbf{F} \, dv = \oint_{\partial V} \mathbf{F} \cdot d\mathbf{s}.$$

Definition 4 (Curl)**.** Let $\mathbf{F}$ be a vector field. In 2 dimensions

$$\nabla \times \mathbf{F} = (\partial_x F_y - \partial_y F_x) \, \hat{\mathbf{z}}.$$

And in 3D

$$\nabla \times \mathbf{F} = \begin{pmatrix} \partial_y F_z - \partial_z F_y \\ \partial_z F_x - \partial_x F_z \\ \partial_x F_y - \partial_y F_x \end{pmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix}.$$

Definition 5 (Curl in curvilinear coordinates)**.** Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a field. In cylindrical coordinates (r, ϕ, z)

$$\begin{aligned} \nabla \times \mathbf{F} = \left(\frac{1}{r} \partial_\phi F_z - \partial_z F_\phi \right) \hat{\mathbf{r}} \\ + (\partial_z F_r - \partial_r F_z) \, \hat{\boldsymbol{\phi}} \\ + \frac{1}{r} \Big[\partial_r (r F_\phi) - \partial_\phi F_r \Big] \hat{\mathbf{z}}, \end{aligned}$$

and in spherical coordinates (r, θ, ϕ)

$$\begin{aligned} \nabla \times \mathbf{F} = \frac{1}{r \sin\theta} \Big[\partial_\theta (\sin\theta F_\phi) - \partial_\phi F_\theta \Big] \hat{\mathbf{r}} \\ + \frac{1}{r} \left[\frac{1}{\sin\theta} \partial_\phi F_r - \partial_r (r F_\phi) \right] \hat{\boldsymbol{\theta}} \\ + \frac{1}{r} \Big[\partial_r (r F_\theta) - \partial_\theta F_r \Big] \hat{\boldsymbol{\phi}}. \end{aligned}$$

Theorem 5 (Stokes' theorem)**.**

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

1.3 Second vector derivatives

Definition 6 (Laplacian operator)**.** A second vector derivative is so important that it has a special name. For a scalar function $f : \mathbb{R}^m \to \mathbb{R}$ the divergence of the gradient

$$\nabla^2 f = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f) = \sum_{i=1}^{m} \partial^2_{x_i} f_{x_i}$$

is called the *Laplacian operator*.

Theorem 6 (Laplacian in curvilinear coordinates)**.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field. In cylindrical coordinates (r, ϕ, z)

$$\nabla^2 f = \frac{1}{r}\partial_r(r\partial_r f) + \frac{1}{r^2}\partial^2_\phi f + \partial^2_z f$$

and in spherical coordinates (r, θ, ϕ)

$$\nabla^2 f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial^2_\phi f.$$

Definition 7 (Vector Laplacian)**.** The Laplacian operator can be extended on a vector field $\mathbf{F}$ to the *Laplacian vector* by applying the Laplacian to each component:

$$\boldsymbol{\nabla}^2 \mathbf{F} = (\nabla^2 F_x)\hat{\mathbf{x}} + (\nabla^2 F_y)\hat{\mathbf{y}} + (\nabla^2 F_z)\hat{\mathbf{z}}.$$

The vector Laplacian can also be defined as

$$\boldsymbol{\nabla}^2 \mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot \mathbf{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times \mathbf{F}).$$

Theorem 7 (Product rules and second derivatives)**.** Let f, g be sufficiently differentiable scalar functions $D \subseteq \mathbb{R}^m \to \mathbb{R}$ and $\mathbf{A}, \mathbf{B}$ be sufficiently differentiable vector fields in $\mathbb{R}^m$ (with $m = 2$ or 3 for equations with the curl).

- Rules with the gradient

$$\begin{aligned}
\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot \mathbf{A}) &= \boldsymbol{\nabla}\times \boldsymbol{\nabla}\times \mathbf{A} + \boldsymbol{\nabla}^2 \mathbf{A}\\
\boldsymbol{\nabla}(f\cdot g) &= (\boldsymbol{\nabla} f)\cdot g + f\cdot \boldsymbol{\nabla} g\\
\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}\\
&\quad + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})
\end{aligned}$$

- Rules with the divergence

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f) &= \nabla^2 f\\
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times \mathbf{A}) &= 0\\
\boldsymbol{\nabla}\cdot(f\cdot \mathbf{A}) &= (\boldsymbol{\nabla} f)\cdot\mathbf{A} + f\cdot(\boldsymbol{\nabla}\cdot\mathbf{A})\\
\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) &= (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})
\end{aligned}$$

- Rules with the curl

$$\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) &= \mathbf{0}\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}^2\mathbf{A}\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}^2\mathbf{A}) &= \boldsymbol{\nabla}^2(\boldsymbol{\nabla}\times\mathbf{A})\\
\boldsymbol{\nabla}\times(f\cdot\mathbf{A}) &= (\boldsymbol{\nabla} f)\times\mathbf{A} + f\cdot\boldsymbol{\nabla}\times\mathbf{A}\\
\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) &= (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}\\
&\quad + \mathbf{A}\cdot(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}\cdot(\boldsymbol{\nabla}\cdot\mathbf{A})
\end{aligned}$$

2 Electrodynamics Recap

2.1 Maxwell's equations

Maxwell's equations in matter in their integral form are

$$\oint_{\partial S} \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}, \tag{1a}$$

$$\oint_{\partial S} \mathbf{H}\cdot d\mathbf{l} = \int_S (\mathbf{J} + \partial_t\mathbf{D})\cdot d\mathbf{s}, \tag{1b}$$

$$\oint_{\partial V} \mathbf{D}\cdot d\mathbf{s} = \int_V \rho\, dv, \tag{1c}$$

$$\oint_{\partial V} \mathbf{B}\cdot d\mathbf{s} = 0. \tag{1d}$$

Where $\mathbf{J}$ and ρ are the *free current density* and *free charge density* respectively.

2.2 Linear materials and boundary conditions

Inside of so called isotropic linear materials fluxes and current densities are proportional and parallel to the fields, i.e.

$$\mathbf{D} = \epsilon\mathbf{E}, \qquad \mathbf{J} = \sigma\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}.$$

Where two materials meet the following boundary conditions must be satisfied:

$$\begin{aligned}
\hat{\mathbf{n}}\cdot\mathbf{D}_1 &= \hat{\mathbf{n}}\cdot\mathbf{D}_2 + \rho_s & \hat{\mathbf{n}}\times\mathbf{E}_1 &= \hat{\mathbf{n}}\times\mathbf{E}_2\\
\hat{\mathbf{n}}\cdot\mathbf{J}_1 &= \hat{\mathbf{n}}\cdot\mathbf{J}_2 - \partial_t\rho_s & \hat{\mathbf{n}}\times\mathbf{H}_1 &= \hat{\mathbf{n}}\times\mathbf{H}_2 + \mathbf{J}_s\\
\hat{\mathbf{n}}\cdot\mathbf{B}_1 &= \hat{\mathbf{n}}\cdot\mathbf{B}_2 - \partial_t\rho_s & \hat{\mathbf{n}}\times\mathbf{M}_1 &= \hat{\mathbf{n}}\times\mathbf{M}_2 + \mathbf{J}_{s,m}
\end{aligned}$$

2.3 Potentials

Because $\mathbf{E}$ is often conservative ($\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$), and $\boldsymbol{\nabla}\cdot\mathbf{B}$ is always zero, it is often useful to use *potentials* to describe these quantities instead. The electric scalar potential and magnetic vector potentials are in their integral form:

$$\varphi = \int_{\mathsf{A}}^{\mathsf{B}} \mathbf{E}\cdot d\mathbf{l}, \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}dv}{R}$$

With differential operators:

$$\mathbf{E} = -\boldsymbol{\nabla}\varphi, \qquad \mu_0\mathbf{J} = -\boldsymbol{\nabla}^2\,\mathbf{A}.$$

By taking the divergence on both sides of the equation with the electric field we get $\rho/\epsilon = -\nabla^2\,\varphi$, which also contains the Laplacian operator. We will study equations with of form in §3.

3 Laplace and Poisson's equations

The so called *Poisson's equation* has the form

$$\nabla^2\,\varphi = -\frac{\rho}{\epsilon}.$$

When the right side of the equation is zero, it is also known as *Laplace's equation.*

3.1 Easy solutions of Laplace and Poisson's equations

3.1.1 Geometry with zenithal and azimuthal symmetries (Übung 2)

Suppose we have a geometry where, using spherical coordinates, there is a symmetry such that the solution does not depend on ϕ or θ. Then Laplace's equation reduces down to

$$\nabla^2\,\varphi = \frac{1}{r^2}\partial_r(r^2\partial_r\varphi) = 0,$$

which has solutions of the form

$$\varphi(r) = \frac{C_1}{r} + C_2.$$

3.2 Geometry with azimuthal and translational symmetry (Übung 3)

Suppose that when using cylindrical coordinates, the solution does not depend on ϕ or z. Then Laplace's equation becomes

$$\nabla^2\,A_z = \frac{1}{r}\partial_r(r\partial_r A_z) = 0.$$