Notes of "Funktionen mehrerer Variablen"

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1 Preface

These are just my personal notes of the FuVar course, and definitively not a rigorously constructed mathematical text.

2 Derivatives of vector valued scalar functions

Definition 1 (Partial derivative)**.** A vector valued function $f : \mathbb{R}^m \to \mathbb{R}$, with $\mathbf{v} \in \mathbb{R}^m$, has a partial derivative with respect to v_i defined as

$$\partial_{v_i} f(\mathbf{v}) = \frac{\partial f}{\partial v_i} = \lim_{h \to 0} \frac{f(\mathbf{v} + h\mathbf{e}_i) - f(\mathbf{v})}{h}$$

Theorem 1. (Schwarz's theorem, symmetry of partial derivatives) Under some generally satisfied conditions (continuity of n-th order partial derivatives) Schwarz's theorem states that it is possible to swap the order of differentiation.

$$\partial_x \partial_y f(x, y) = \partial_y \partial_x f(x, y)$$

Application 1 (Find the slope of an implicit curve)**.** Let $f(x, y) = 0$ be an implicit curve. Its slope at any point where $\partial_y f \neq 0$ is $m = -\partial_x f / \partial_y f$

Definition 2 (Total differential)**.** The total differential df of $f : \mathbb{R}^m \to \mathbb{R}$ is

$$df = \sum_{i=1}^{m} \partial_{x_i} f \cdot dx.$$

That reads, the *total* change is the sum of the change in each direction. This implies

$$\frac{df}{dx_k} = \frac{\partial f}{\partial x_k} + \sum_{i \in \{1 \leq i \leq m : i \neq k\}} \frac{\partial f}{\partial x_i} \cdot \frac{dx_i}{dx_k},$$

i.e. the change in direction x_k is how f changes in x_k (ignoring other directions) plus, how f changes with respect to each other variable x_i times how they (x_i) change with respect to x_k.

Application 2 (Linearization)**.** A function $f : \mathbb{R}^m \to \mathbb{R}$ has a linearization g at $\mathbf{x}_0$ given by

$$g(\mathbf{x}) = f(\mathbf{x}_0) + \sum_{i=1}^{m} \partial_{x_i} f(\mathbf{x}_0)(x_i - x_{i,0}),$$

if all partial derivatives are defined at $\mathbf{x}_0$. With the gradient (defined below) $g(\mathbf{x}) = f(\mathbf{x}_0) + \boldsymbol{\nabla} f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$.

Application 3 (Propagation of uncertanty)**.** Given a measurement of m values in a vector $\mathbf{x} \in \mathbb{R}^m$ with values given in the form $x_i = \bar{x}_i \pm \sigma_{x_i}$, a linear approximation of the error of a dependent variable $y = f(\mathbf{x})$ is computed with

$$y = \bar{y} \pm \sigma_y \approx f(\bar{\mathbf{x}}) \pm \sqrt{\sum_{i=1}^{m} \left(\partial_{x_i} f(\bar{\mathbf{x}}) \sigma_{x_i}\right)^2}$$

Definition 3 (Gradient vector)**.** The *gradient* of a function $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^m$ is a column vector[1] containing the partial derivatives in each direction.

$$\boldsymbol{\nabla} f(\mathbf{x}) = \sum_{i=1}^{m} \partial_{x_i} f(\mathbf{x}) \mathbf{e}_i = \begin{pmatrix} \partial_{x_1} f(\mathbf{x}) \\ \vdots \\ \partial_{x_m} f(\mathbf{x}) \end{pmatrix}$$

Theorem 2. The gradient vector always points towards *the direction of steepest ascent*, and thus is always perpendicular to contour lines.

Definition 4 (Directional derivative)**.** A function $f(\mathbf{x})$ has a directional derivative in direction $\mathbf{v}$ (with $|\mathbf{v}| = 1$) of

$$\frac{\partial f}{\partial \mathbf{v}} = \nabla_{\mathbf{v}} f = \mathbf{v} \cdot \boldsymbol{\nabla} f = \sum_{i=1}^{m} v_i \partial_{x_i} f$$

Definition 5 (Jacobian Matrix)**.** The *Jacobian* $\mathbf{J}_f$ (sometimes written as $\frac{\partial(f_1, \ldots f_m)}{\partial(x_1, \ldots, x_n)}$) of a function $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$ is a matrix $\in \mathbb{R}^{m \times n}$ whose entry at the i-th row and j-th column is given by $(\mathbf{J}_f)_{i,j} = \partial_{x_j} f_i$, so

$$\mathbf{J}_f = \begin{pmatrix} \partial_{x_1} f_1 & \cdots & \partial_{x_m} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_n & \cdots & \partial_{x_m} f_n \end{pmatrix} = \begin{pmatrix} (\boldsymbol{\nabla} f_1)^t \\ \vdots \\ (\boldsymbol{\nabla} f_m)^t \end{pmatrix}$$

Remark 1. In the scalar case ($n = 1$) the Jacobian matrix is the transpose of the gradient vector.

Definition 6 (Hessian matrix)**.** Given a function $f : \mathbb{R}^m \to \mathbb{R}$, the square matrix whose entry at the i-th row and j-th column is the second derivative of f first with respect to x_j and then to x_i is known as the *Hessian* matrix. $(\mathbf{H}_f)_{i,j} = \partial_{x_i} \partial_{x_j} f$ or

$$\mathbf{H}_f = \begin{pmatrix} \partial_{x_1} \partial_{x_1} f & \cdots & \partial_{x_1} \partial_{x_m} f \\ \vdots & \ddots & \vdots \\ \partial_{x_m} \partial_{x_1} f & \cdots & \partial_{x_m} \partial_{x_m} f \end{pmatrix}$$

Because (almost always) the order of differentiation does not matter, it is a symmetric matrix.

[1] In matrix notation it is also often defined as row vector to avoid having to do some transpositions in the Jacobian matrix and dot products in directional derivatives

3 Methods for maximization and minimization problems

3.1 Analytical methods

Method 1 (Find stationary points)**.** Given a function $f : D \subseteq \mathbb{R}^m \to \mathbb{R}$, to find its maxima and minima we shall consider the points

- that are on the boundary[2] of the domain ∂D,
- where the gradient $\boldsymbol{\nabla} f$ is not defined,
- that are stationary, i.e. where $\boldsymbol{\nabla} f = \mathbf{0}$.

Method 2 (Determine the type of stationary point for 2 dimensions)**.** Given a scalar function of two variables $f(x, y)$ and a stationary point $\mathbf{x}_s$ (where $\boldsymbol{\nabla} f(\mathbf{x}_s) = \mathbf{0}$), we define the *discriminant*

$$\Delta = \partial_x^2 f \partial_y^2 f - \partial_y \partial_x f$$

- if $\Delta > 0$ then $\mathbf{x}_s$ is an extrema, if $\partial_x^2 f(\mathbf{x}_s) < 0$ it is a maximum, whereas if $\partial_x^2 f(\mathbf{x}_s) > 0$ it is a minimum;
- if $\Delta < 0$ then $\mathbf{x}_s$ is a saddle point;
- if $\Delta = 0$ we need to analyze further.

Remark 2. The previous method is obtained by studying the second directional derivative $\nabla_{\mathbf{v}} \nabla_{\mathbf{v}} f$ at the stationary point in direction of a vector $\mathbf{v} = \mathbf{e}_1 \cos(\alpha) + \mathbf{e}_2 \sin(\alpha)$.

Method 3 (Determine the type of stationary point in higher dimensions)**.** Given a scalar function of multiple variables $f(\mathbf{x})$ and a stationary point $\mathbf{x}_s$ ($\boldsymbol{\nabla} f(\mathbf{x}_s) = \mathbf{0}$), we compute the Hessian matrix $\mathbf{H}_f(\mathbf{x}_s)$ and its eigenvalues $\lambda_1, \ldots, \lambda_m$, then

- if all $\lambda_i > 0$, the point is a minimum;
- if all $\lambda_i < 0$, the point is a maximum;
- if there are both positive and negative eigenvalues, it is a saddle point.

In the other cases, when there are $\lambda_i \leq 0$ and/or $\lambda_i \geq 0$ further analysis is required.

Remark 3. Recall that to compute the eigenvalues of a matrix, one must solve the equation $(\mathbf{H} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$. Which can be done by solving the characteristic polynomial $\det(\mathbf{H} - \lambda \mathbf{I}) = 0$ to obtain $\dim(\mathbf{H})$ λ_i, which when plugged back in result in a overdetermined system of equations.

Method 4 (Quickly find the eigenvalues of a 2×2 matrix)**.** This is a nice trick. For a square matrix $\mathbf{H}$, let

$$m = \frac{1}{2} \operatorname{tr} \mathbf{H} = \frac{a + d}{2}, \quad p = \det \mathbf{H} = ad - bc,$$

then $\lambda_{1,2} = m \pm \sqrt{m^2 - p}$.

[2] If it belongs to f.

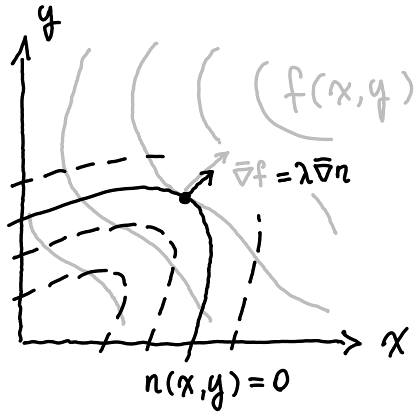


Figure 1: Intuition for the method of Lagrange multipliers. Extrema of a constrained function are where $\boldsymbol{\nabla} f$ is proportional to $\boldsymbol{\nabla} n$.

Method 5 (Search for a constrained extremum in 2 dimensions)**.** Let $n(x, y) = 0$ be a constraint in the search of the extrema of a function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$. To find the extrema we look for points

- on the boundary[2] $\mathbf{u} \in \partial D$ where $n(\mathbf{u}) = 0$;
- $\mathbf{u}$ where the gradient either does not exist or is $\mathbf{0}$, and satisfy $n(\mathbf{u}) = 0$;
- that solve the system of equations

$$\begin{cases} \partial_x f(\mathbf{u}) \cdot \partial_y n(\mathbf{u}) = \partial_y f(\mathbf{u}) \cdot \partial_x n(\mathbf{u}) \\ n(\mathbf{u}) = 0 \end{cases}$$

Method 6 (Search for a constrained extremum in higher dimensions, method of Lagrange multipliers)**.** We wish to find the extrema of $f : D \subseteq \mathbb{R}^m \to \mathbb{R}$ under $k < m$ constraints $n_1 = 0, \cdots, n_k = 0$. To find the extrema we consider the following points:

- Points on the boundary[2] $\mathbf{u} \in \partial D$ that satisfy $n_i(\mathbf{u}) = 0$ for all $1 \leq i \leq k$,
- Points $\mathbf{u} \in D$ where either
  - any of $\boldsymbol{\nabla} f, \boldsymbol{\nabla} n_1, \ldots, \boldsymbol{\nabla} n_k$ do not exist, or
  - $\boldsymbol{\nabla} n_1, \ldots, \boldsymbol{\nabla} n_k$ are linearly *dependent*,

  and that satisfy $0 = n_1(\mathbf{u}) = \ldots = n_k(\mathbf{u})$.
- Points that solve the system of $m + k$ equations

$$\begin{cases} \boldsymbol{\nabla} f(\mathbf{u}) = \sum_{i=1}^{k} \lambda_i \boldsymbol{\nabla} n_i(\mathbf{u}) & (m\text{-dimensional}) \\ n_i(\mathbf{u}) = 0 & \text{for } 1 \leq i \leq k \end{cases}$$

  The λ values are known as *Lagrange multipliers*.

The calculation of the last point can be written more compactly by defining the *Lagrangian*

$$\mathcal{L}(\mathbf{u}, \boldsymbol{\lambda}) = f(\mathbf{u}) - \sum_{i=0}^{k} \lambda_i n_i(\mathbf{u}),$$

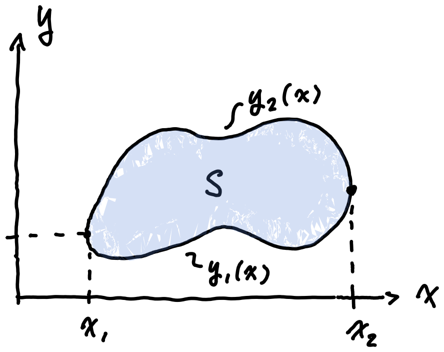


Figure 2: Double integral.

where $\boldsymbol{\lambda} = \lambda_1, \dots, \lambda_k$ and then solving the $m+k$ dimensional equation $\boldsymbol{\nabla}\mathcal{L}(\mathbf{u}, \boldsymbol{\lambda}) = \mathbf{0}$ (this is generally used in numerical computations and not very useful by hand).

3.2 Numerical methods

Method 7 (Newton's method)**.** For a function $f : \mathbb{R}^m \to \mathbb{R}$ we wish to numerically find its stationary points (where $\boldsymbol{\nabla} f = \mathbf{0}$).

1. Pick a starting point $\mathbf{x}_0$.
2. Set the linearisation[3] of $\boldsymbol{\nabla} f$ at $\mathbf{x}_k$ to zero and solve for $\mathbf{x}_{k+1}$.

$$\boldsymbol{\nabla} f(\mathbf{x}_k) + \mathbf{H}_f(\mathbf{x}_k)(\mathbf{x}_{k+1} - \mathbf{x}_k) = \mathbf{0}$$
$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{H}_f^{-1}(\mathbf{x}_k)\boldsymbol{\nabla} f(\mathbf{x}_k)$$

3. Repeat the last step until the magnitude of the error $|\boldsymbol{\epsilon}| = |\mathbf{H}_f^{-1}(\mathbf{x}_k)\boldsymbol{\nabla} f(\mathbf{x}_k)|$ is sufficiently small.

Method 8 (Gradient ascent / descent)**.** Given $f : \mathbb{R}^m \to \mathbb{R}$ we wish to numerically find the stationary points (where $\boldsymbol{\nabla} f = \mathbf{0}$).

1. Define an arbitrarily small length η and a starting point $\mathbf{x}_0$
2. Compute $\mathbf{v} = \pm\boldsymbol{\nabla} f(\mathbf{x}_k)$ (positive for ascent, negative for descent), then $\mathbf{x}_{k+1} = \mathbf{x}_k + \eta\mathbf{v}$ if the rate of change ϵ is acceptable ($\epsilon = |\boldsymbol{\nabla} f(\mathbf{x}_{k+1})| > 0$) else recompute $\mathbf{v} := \pm\boldsymbol{\nabla} f(\mathbf{x}_{k+1})$.
3. Stop when the rate of change ϵ stays small enough for many iterations.

4 Integration of vector valued scalar functions

Theorem 3 (Change the order of integration for double integrals)**.** For a double integral over a region S (see Fig. 2) we need to compute

$$\iint_S f(x,y)\,ds = \int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)} f(x,y)\,dydx.$$

[3]The gradient becomes a hessian matrix.

If $y_1(x)$ and $y_2(x)$ are bijective we can swap the order of integration by finding the inverse functions $x_1(y)$ and $x_2(y)$. If they are not bijective (like in Fig. 2), the region must be split into smaller parts. If the region is a rectangle it is always possible to change the order of integration.

Theorem 4 (Transformation of coordinates in 2 dimensions)**.** Given two "nice" functions $x(u,v)$ and $y(u,v)$, that means are a bijection from S to S' with continuous partial derivatives and nonzero Jacobian determinant $|\mathbf{J}| = \partial_u x \partial_v y - \partial_v x \partial_u y$, which transform the coordinate system. Then

$$\iint_S f(x,y)\,ds = \iint_{S'} f(x(u,v), y(u,v))|\mathbf{J}|\,ds.$$

Theorem 5 (Transformation of coordinates)**.** The generalization of theorem 4 is quite simple. For an m-integral of a function $f : \mathbb{R}^m \to \mathbb{R}$ over a region B, we let $\mathbf{g}(\mathbf{u})$ be "nice" functions that transform the coordinate system. Then as before

$$\int_B f(\mathbf{r})\,ds = \int_{B'} f(\mathbf{g}(\mathbf{u}))|\mathbf{J}_\mathbf{g}|\,ds.$$

Application 4 (Physics)**.** Given the mass m and density function ρ of an object, its *center of mass* is calculated with

$$\mathbf{x}_c = \frac{1}{m}\int_V \rho(\mathbf{r})\mathbf{r}\,dv \overset{\rho\text{ const.}}{=} \frac{1}{V}\int_V \mathbf{r}\,dv.$$

The (scalar) *moment of inertia* J of an object is given by

$$J = \int_V \rho(\mathbf{r})r^2\,dv.$$

5 Parametric curves, line and surface integrals

Definition 7 (Parametric curve)**.** A parametric curve is a vector function $\mathcal{C} : \mathbb{R} \to W \subseteq \mathbb{R}^n, t \mapsto \mathbf{f}(t)$, that takes a parameter t.

Theorem 6 (Derivative of a curve)**.** The derivative of a curve is

$$\begin{aligned}\mathbf{f}'(t) &= \lim_{h\to 0}\frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h}\\ &= \sum_{i=0}^{n}\left(\lim_{h\to 0}\frac{f_i(t+h) - f_i(t)}{h}\right)\mathbf{e}_i\\ &= \sum_{i=0}^{n}\frac{df_i}{dt}\mathbf{e}_i = \left(\frac{df_1}{dt}, \dots, \frac{df_m}{dt}\right)^t.\end{aligned}$$

Theorem 7 (Multivariable chain rule)**.** Let $\mathbf{x} : \mathbb{R} \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$, so that $f \circ \mathbf{x} : \mathbb{R} \to \mathbb{R}$, then the multivariable chain rule states:

$$\frac{d}{dt}f(\mathbf{x}(t)) = \boldsymbol{\nabla} f(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = \nabla_{\mathbf{x}'(t)} f(\mathbf{x}(t)).$$

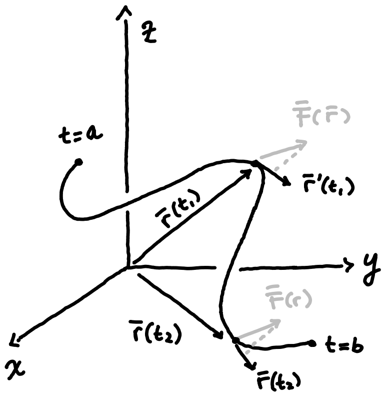


Figure 3: Line integral in a vector field.

Theorem 8 (Signed area enclosed by a planar parametric curve)**.** A planar (2D) parametric curve $(x(t), y(t))^t$ with $t \in [r, s]$ that does not intersect itself encloses a surface with area

$$A = \int_r^s x'(t)y(t)\,dt = \int_r^s x(t)y'(t)\,dt.$$

Definition 8 (Line integral in a scalar field)**.** Let $\mathcal{C} : [a, b] \to \mathbb{R}^n, t \mapsto \mathbf{r}(t)$ be a parametric curve. The *line integral* in a field $f(\mathbf{r})$ is the integral of the signed area under the curve traced in $\mathbb{R}^n$, and is computed with

$$\int_\mathcal{C} f(\mathbf{r})\,d\ell = \int_\mathcal{C} f(\mathbf{r})\,|d\mathbf{r}| = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt.$$

Application 5 (Length of a parametric curve)**.** By computing the line integral of the function $1(\mathbf{r})$ we get the length of the parametric curve $\mathcal{C} : [a, b] \to \mathbb{R}^n$.

$$\int_\mathcal{C} d\ell = \int_\mathcal{C} |d\mathbf{r}| = \int_a^b \sqrt{\sum_{i=1}^n x_i'(t)^2}\,dt$$

The special case with the scalar function $f(x)$ results in $\int_a^b \sqrt{1 + f'(x)^2}\,dx$.

Definition 9 (Line integral in a vector field)**.** The line integral in a vector field $\mathbf{F}(\mathbf{r})$ is the "sum" of the projections of the field's vectors on the tangent of the parametric curve $\mathcal{C}$.

$$\int_\mathcal{C} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

Theorem 9 (Line integral in the opposite direction)**.** By integrating while moving backwards $(-t)$ on the parametric curve gives

$$\int_{-\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = -\int_\mathcal{C} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}.$$

Definition 10 (Conservative field)**.** A vector field is said to be *conservative* the line integral over a closed path is zero.

$$\oint_\mathcal{C} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$$

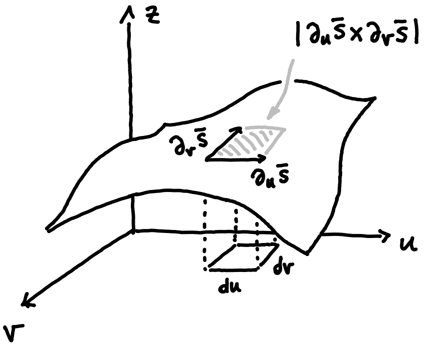


Figure 4: Surface integral.

Theorem 10. For a twice partially differentiable vector field $\mathbf{F}$ in n dimensions without "holes", i.e. in which each closed curve can be contracted to a point (simply connected open set), the following statements are equivalent:

- $\mathbf{F}$ is conservative,
- $\mathbf{F}$ is path-independent,
- $\mathbf{F}$ is a *gradient field*, i.e. there is a function ϕ called *potential* such that $\mathbf{F} = \boldsymbol{\nabla}\phi$,
- $\mathbf{F}$ satisfies the condition $\partial_{x_j} F_i = \partial_{x_i} F_j$ for all $i, j \in \{1, 2, \ldots, n\}$. In the 2D case $\partial_x F_y = \partial_y F_x$, and in 3D
$$\begin{cases} \partial_y F_x = \partial_x F_y \\ \partial_z F_y = \partial_y F_z \\ \partial_x F_z = \partial_z F_x \end{cases}$$

Theorem 11. In a conservative field $\mathbf{F}$ with gradient ϕ, using the multivariable the chain rule:

$$\begin{aligned} \int_\mathcal{C} \mathbf{F} \cdot d\mathbf{r} &= \int_\mathcal{C} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\ &= \int_\mathcal{C} \boldsymbol{\nabla}\phi(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\ &= \int_\mathcal{C} \frac{d\phi(\mathbf{r}(t))}{dt}\,dt = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)). \end{aligned}$$

Definition 11 (Parametric surface)**.** A parametric surface is a vector function $\mathcal{S} : W \subseteq \mathbb{R}^2 \to \mathbb{R}^3$.

Theorem 12 (Area of a parametric surface)**.** The area spanned by a parametric surface $\mathbf{s}(u, v)$, with continuous partial derivatives and that satisfy $\partial_u \mathbf{s} \times \partial_v \mathbf{s} \neq \mathbf{0}$, is given by

$$A = \int_\mathcal{S} ds = \iint |\partial_u \mathbf{s} \times \partial_v \mathbf{s}|\,dudv.$$

Definition 12 (Scalar surface integral)**.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function on a parametric surface $\mathbf{s} : W \subseteq \mathbb{R}^2 \to \mathbb{R}^3$. The surface integral of f over $\mathcal{S}$ is

$$\int_\mathcal{S} f\,ds = \iint_W f(\mathbf{s}(u, v)) \cdot |\partial_u \mathbf{s} \times \partial_v \mathbf{s}|\,dudv.$$

	Volume dv	Surface $d\mathbf{s}$
Cartesian	–	$dx\,dy$
Polar	–	$r d\, r d\phi$
Curvilinear	–	$\lvert \mathbf{J}_f \rvert \, du\, dv$
Cartesian	$dx\,dy\,dz$	$\hat{\mathbf{z}}\, dx\, dy$
Cylindrical	$r\, dr\, d\phi\, dz$	$\hat{\mathbf{z}} r\, dr\, d\phi$
		$\hat{\boldsymbol{\phi}}\, dr\, dz$
		$\hat{\mathbf{r}} r\, d\phi\, dz$
Spherical	$r^2 \sin\theta\, dr\, d\theta\, d\phi$	$\hat{\mathbf{r}} r^2 \sin\theta\, d\theta\, d\phi$
Curvilinear	$\lvert \mathbf{J}_f \rvert \, du\, dv\, dw$	–

Table 1: Differential elements for integration.

6 Vector analysis

Definition 13 (Flux)**.** In a vector field $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$ we define the *flux* through a parametric surface $\mathcal{S}$ as

$$\Phi = \int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds.$$

If $\mathcal{S}$ is a closed surface we write $\mathring{\Phi} = \oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{s}$.

If we now take the normalized flux on the surface of an arbitrarily small volume V (limit as $V \to 0$) we get the *divergence*

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \lim_{V \to 0} \frac{1}{V} \oint_{\partial V} \mathbf{F} \cdot d\mathbf{s}.$$

Theorem 13 (Formula for divergence)**.** Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$ be a vector field. The divergence of $\mathbf{F} = (F_{x_1}, \ldots, F_{x_m})^t$ is

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \sum_{i=1}^{m} \partial_{x_i} F_{x_i},$$

as suggested by the (ab)use of the dot product notation.

Theorem 14 (Divergence theorem, Gauss's theorem)**.** Because the flux on the boundary ∂V of a volume V contains information of the field inside of V, it is possible relate the two with

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F}\, dv = \oint_{\partial V} \mathbf{F} \cdot d\mathbf{s}.$$

Definition 14 (Circulation, Vorticity)**.** The result of a closed line integral can be interpreted as a macroscopic measure how much the field rotates around a given point, and is thus sometimes called *circulation* or *vorticity*.

As before, if we now make the area A enclosed by the parametric curve for the circulation arbitrarily small, normalize it, and use Gauss's theorem we get a local measure called *curl*.

$$\boldsymbol{\nabla} \times \mathbf{F} = \lim_{A \to 0} \frac{\hat{\mathbf{n}}}{A} \oint_{\partial A} \mathbf{F} \cdot d\mathbf{s}$$

Notice that the curl is a vector, normal to the enclosed surface A.

Theorem 15 (Formula for curl)**.** Let $\mathbf{F}$ be a vector field. In 2 dimensions

$$\boldsymbol{\nabla} \times \mathbf{F} = (\partial_x F_y - \partial_y F_x)\, \hat{\mathbf{z}}.$$

And in 3D

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{pmatrix} \partial_y F_z - \partial_z F_y \\ \partial_z F_x - \partial_x F_z \\ \partial_x F_y - \partial_y F_x \end{pmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix}.$$

Theorem 16 (Stokes' theorem)**.**

$$\int_{\mathcal{S}} \boldsymbol{\nabla} \times \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial \mathcal{S}} \mathbf{F} \cdot d\mathbf{r}$$

Theorem 17 (Green's theorem)**.** The special case of Stokes' theorem in 2D is known as Green's theorem.

$$\int_{\mathcal{S}} \partial_x F_y - \partial_y F_x\, ds = \oint_{\partial \mathcal{S}} \mathbf{F} \cdot d\mathbf{r}$$

Definition 15 (Laplacian operator)**.** A second vector derivative is so important that it has a special name. For a scalar function $f : \mathbb{R}^m \to \mathbb{R}$ the divergence of the gradient

$$\nabla^2 = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = \sum_{i=1}^{m} \partial_{x_i}^2 f_{x_i}$$

is called the *Laplacian operator*.

Definition 16 (Vector Laplacian)**.** The Laplacian operator can be extended on a vector field $\mathbf{F}$ to the *Laplacian vector* by applying the Laplacian to each component:

$$\boldsymbol{\nabla}^2 \mathbf{F} = (\nabla^2 F_x)\hat{\mathbf{x}} + (\nabla^2 F_y)\hat{\mathbf{y}} + (\nabla^2 F_z)\hat{\mathbf{z}}.$$

The vector Laplacian can also be defined as

$$\boldsymbol{\nabla}^2 \mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{F}).$$

Theorem 18 (Product rules and second derivatives)**.** Let f, g be sufficiently differentiable scalar functions $D \subseteq \mathbb{R}^m \to \mathbb{R}$ and $\mathbf{A}, \mathbf{B}$ be sufficiently differentiable vector fields in $\mathbb{R}^m$ (with $m = 2$ or 3 for equations with the curl).

- Rules with the gradient

$$\begin{aligned} \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) &= \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla}^2 \mathbf{A} \\ \boldsymbol{\nabla}(f \cdot g) &= (\boldsymbol{\nabla} f) \cdot g + f \cdot \boldsymbol{\nabla} g \\ \boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} \\ &\quad + \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) \end{aligned}$$

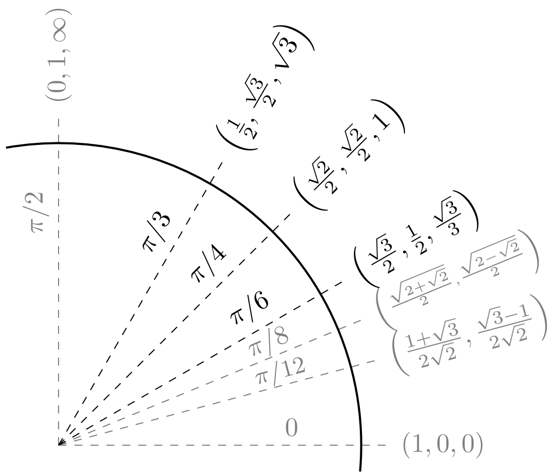
- Rules with the divergence

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) &= \nabla^2 f \\ \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) &= 0 \\ \boldsymbol{\nabla} \cdot (f \cdot \mathbf{A}) &= (\boldsymbol{\nabla} f) \cdot \mathbf{A} + f \cdot (\boldsymbol{\nabla} \cdot \mathbf{A}) \\ \boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) &= (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) \end{aligned}$$

- Rules with the curl

$$\begin{aligned}
\nabla\times(\nabla f) &= \mathbf{0}\\
\nabla\times(\nabla\times \mathbf{A}) &= \nabla(\nabla\cdot \mathbf{A}) - \nabla^2 \mathbf{A}\\
\nabla\times(\nabla^2 \mathbf{A}) &= \nabla^2(\nabla\times \mathbf{A})\\
\nabla\times(f\cdot \mathbf{A}) &= (\nabla f)\times \mathbf{A} + f\cdot\nabla\times \mathbf{A}\\
\nabla\times(\mathbf{A}\times \mathbf{B}) &= (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}\\
&\quad + \mathbf{A}\cdot(\nabla\cdot \mathbf{B}) - \mathbf{B}\cdot(\nabla\cdot \mathbf{A})
\end{aligned}$$

A Trigonometry



$$\cos^2(x) + \sin^2(x) = 1 \qquad \cosh^2(x) - \sinh^2(x) = 1$$

$\cos(\alpha+2\pi)$	$=$	$\cos(\alpha)$	$\sin(\alpha+2\pi)$	$=$	$\sin(\alpha)$
$\cos(-\alpha)$	$=$	$\cos(\alpha)$	$\sin(-\alpha)$	$=$	$-\sin(\alpha)$
$\cos(\pi-\alpha)$	$=$	$-\cos(\alpha)$	$\sin(\pi-\alpha)$	$=$	$\sin(\alpha)$
$\cos(\frac{\pi}{2}-\alpha)$	$=$	$\sin(\alpha)$	$\sin(\frac{\pi}{2}-\alpha)$	$=$	$\cos(\alpha)$
$\cos(\alpha+\beta)$	$=$	$\cos\alpha\cos\beta - \sin\alpha\sin\beta$			
$\sin(\alpha+\beta)$	$=$	$\sin\alpha\cos\beta - \cos\alpha\sin\beta$			
$\cos(2\alpha)$	$=$	$\cos^2\alpha - \sin^2\alpha$			
	$=$	$1 - 2\sin^2\alpha$			
	$=$	$2\cos^2\alpha - 1$			
$\sin(2\alpha)$	$=$	$2\sin\alpha\cos\alpha$			
$\tan(2\alpha)$	$=$	$(2\tan\alpha)(1+\tan^2\alpha)^{-1}$			

B Derivative

Let f, u, v be differentiable functions of x.

$$(af)' = af' \qquad (u(v))' = u'(v)v'$$

$$(uv)' = u'v + uv' \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\left(\sum u_i\right)' = \sum u_i' \qquad (\ln u)' = \frac{u'}{u}$$

$$(f^{-1})' = \frac{1}{f'(f^{-1}(x))}$$

C Integration

Let f, u, v be integrable functions of x.

Linearity	$\int k(u+v) = k\left(\int u + \int v\right)$
Partial fraction decomposition	$\int \frac{Q}{P_n}\,dx = \sum_{k=1}^{n}\int \frac{A_k}{x-r_k}\,dx$
Affine transformation	$\int f(\lambda x + \ell)\,dx = \frac{1}{\lambda}F(\lambda x + \ell) + C$
Integration by parts	$\int u\,dv = uv - \int v\,du$
Power rule $n \neq -1$	$\int u^n \cdot u' = \frac{u^{n+1}}{n+1} + C$
Logarithm rule	$\int \frac{u'}{u} = \ln\lvert u\rvert + C$
General substitution $x = g(u)$	$\int f(x)\,dx = \int (f\circ g)\; g'\,du$ $= \int \frac{f\circ g}{(g^{-1})'\circ g}\,du$
Universal substitution	$t = \tan(x/2), dx = 2/(1+t^2)dt$ $\sin(x) = \frac{2t}{1+t^2},\quad \cos(t) = \frac{1-t^2}{1+t^2}$

D Tables

Some useful derivatives and integrals:

f	f'	f	f'
x^n	nx^{n-1}	a^x	$a^x \ln a$
$\sqrt[n]{x}$	$1/\left(x^n\sqrt[n]{x^{n-1}}\right)$	$\ln x$	$1/x$
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$1/\cos^2 x$	$1/\tan x$	$-1/\sin^2 x$
$\arcsin x$	$1/\sqrt{1-x^2}$	$\arccos x$	$-1/\sqrt{1-x^2}$
$\arctan x$	$1/\left(1+x^2\right)$		
$\sinh x$	$\cosh x$	$\tanh x$	$1/\cosh^2 x$
$\operatorname{arcsinh} x$	$1/\sqrt{1+x^2}$	$\operatorname{arccosh} x$	$1/\sqrt{x^2-1}$

$$\int \ln x\,dx = x\ln x - x + C$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) + C$$

$$\int x^2e^{ax}\,dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) + C$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$$

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