# 1 Complex Numbers

**Definition 1** (Complex Unit and Zero)**.**

$$j \overset{\text{def.}}{=} +\sqrt{-1} \iff j^2 = -1$$

$$1 = (1,0) \quad 0 = (0,0) \quad j = (0,1)$$

**Definition 2** (Negation and Sum)**.** Let $z, w \in \mathbb{C}$

$$-z = (-z_1, -z_2) \quad z \oplus w = (z_1 + w_1, z_2 + w_2)$$

**Lemma 1.** The complex numbers form an additive group. Let $z, w, v \in \mathbb{C}$, we have

$$\begin{aligned} \textit{Identity} \quad & z + 0 = z \\ \textit{Commutativity} \quad & z + w = w + z \\ \textit{Associativity} \quad & z + (w + v) = (z + w) + v \\ \textit{Inverse property} \quad & z + (-z) = (-z) + z = 0 \end{aligned}$$

**Definition 3** (Multiplication)**.** Let $z, w \in \mathbb{C}$

$$(a,b) \odot (c,d) = (ac - bd, ad + bc)$$

**Lemma 2.** The complex numbers form a commutative ring. Let $z, w, v \in \mathbb{C}$

$$\begin{aligned} \textit{Identity} \quad & 1 \cdot z = z \\ \textit{Commutativity} \quad & z \cdot w = w \cdot z \\ \textit{Associativity} \quad & z(wv) = (zw)v \\ \textit{Distributivity} \quad & z(w + v) = zw + zv \end{aligned}$$

**Definition 4** (Real and imaginary part and conjugation)**.** Let $z = a + jb$. The *real* part of $z$ is $\operatorname{Re}(z) = a$, similarly the *imaginary* part is $\operatorname{Im}(z) = b$. We can thus define the *complex conjugate* $\overline{z}$ of $z$ to be

$$z = \operatorname{Re}(z) + j \operatorname{Im}(z) \quad \overline{z} = \operatorname{Re}(z) - j \operatorname{Im}(z)$$

**Definition 5** (Absolute value)**.** If $z = a + jb$ we define the *absolute value* $|z| = \sqrt{a^2 + b^2}$

**Lemma 3** (Properties of absolute value)**.** Let $z, w \in \mathbb{C}$. We have $z\overline{z} = |z|^2$ and as a consequence $|zw| = |z| \cdot |w|$ and $|\overline{z}| = |z|$. In addition we have the inequalities

$$\begin{aligned} -|z| \le \operatorname{Re}(z) \le |z| \qquad & |z| \le |\operatorname{Re}(z)| + |\operatorname{Im}(z)| \\ -|z| \le \operatorname{Im}(z) \le |z| \qquad & |z + w| \le |z| + |w| \end{aligned}$$

The last one is the *triangle inequality.* Notice that $|z| \in \mathbb{R}_0^+$.

**Definition 6** (Reciprocal and quotients)**.** If $z$ is a non-zero complex number we define the *reciprocal* $z^{-1}$ of $z$ to be $z^{-1} = |z|^{-2}\overline{z}$. If $z = 0$ the reciprocal $0^{-1}$ is left undefined. It is now possible to define $z/w = zw^{-1}$ with $z, w \in \mathbb{C}$ and $w \neq 0$.

**Lemma 4** (Properties of conjugation)**.** Let $z, w \in \mathbb{C}$. $\overline{z} = z$ iff $z \in \mathbb{R}$ and $\overline{z} = \overline{w}$ iff $z = w$. Furthermore:

$$\begin{aligned} \overline{\overline{z}} = z \qquad & \overline{z \pm w} = \overline{z} \pm \overline{w} \qquad \operatorname{Re}(z) = (z + \overline{z})/2 \\ \overline{z \cdot w} = \overline{z} \cdot \overline{w} \qquad & \overline{z/w} = \overline{z}/\overline{w} \qquad \operatorname{Im}(z) = (z - \overline{z})/2j \end{aligned}$$

**Definition 7** (Argument and polar notation)**.** An alternative representation of a complex number $z = a + jb$ is its *polar form* $z = r\angle\phi$, where $r = |z|$ and $\phi = \arg z$.

$$a = r\cos\phi \qquad b = r\sin\phi \qquad r = \sqrt{z\overline{z}}$$

For $a = 0$ we define $\phi = \lim_{a \to 0} \arctan(b/a) = \pm\pi/2$ and otherwise

$$\begin{aligned} \phi = \arg(z) &= \begin{cases} \arctan(b/a) & a > 0 \\ \arctan(b/a) + \pi & a < 0 \end{cases} \\ &= \begin{cases} \arccos(a/r) & b \ge 0 \\ -\arccos(b/r) & b < 0 \end{cases} \end{aligned}$$

Another variant of this notation is

$$z = r \operatorname{cjs}\phi = r(\cos\phi + j\sin\phi)$$

**Lemma 5** (Arithmetic in polar notation)**.** Let $z, w \in \mathbb{C}$ then the product $zw$ has

$$|zw| = |z| \cdot |w| \quad \arg(zw) = \arg z + \arg w$$

Similarly the quotient $z/w$ follows

$$|z/w| = |z|/|w| \quad \arg(z/w) = \arg z - \arg w$$

Lastly from the product we see that for $k \in \mathbb{N}$

$$|z^k| = |z|^k \quad \arg z^k = k \arg z$$

**Theorem 1** (De Moivre's formula)**.** Let $n \in \mathbb{N}$

$$(\cos\phi + j\sin\phi)^n = \cos(n\phi) + j\sin(n\phi)$$

As a consequence with the binomial formula $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^n$, recalling that $\binom{n}{k} = n!/(k!(n-k)!)$ (Pascal's triangle), we have

$$\sin(nx) = \sum_{k=0}^{n} \binom{n}{k} (\cos x)^k \, (\sin x)^{n-k} \, \sin\frac{(n-k)\pi}{2}$$

$$\cos(nx) = \sum_{k=0}^{n} \binom{n}{k} (\cos x)^k \, (\sin x)^{n-k} \, \cos\frac{(n-k)\pi}{2}$$

# 2 Complex valued functions

**Definition 8** (Function in $\mathbb{C}$)**.** Let $f : \mathbb{D} \to \mathbb{W}$ with both $\mathbb{D}, \mathbb{W} \subseteq \mathbb{C}$ that maps $z = (a + jb) \mapsto w = (u + jv)$, then $u = \operatorname{Re} f(z)$ and $v = \operatorname{Im} f(z)$. If $f$ is a bijection with inverse $f^{-1}$, then $a = \operatorname{Re} f^{-1}(w), b = \operatorname{Im} f^{-1}(w)$.

**Definition 9** (Differentiation in $\mathbb{C}$)**.** Let $f$ be a function of $z$ and $h \in \mathbb{C}$. We have the limit

$$\lim_{|h| \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

to define the *derivative* of $f$ at the point $z_0$.

**Lemma 6** (Local dilation and rotation)**.** Let $f$ be a differentiable function in $\mathbb{C}$. If $f'(z) \neq 0$ everywhere, then $f$ is a conformal map (i.e. preserves angles) with local dilation of $|f'(z)|$ and rotation of $\arg f'(z)$

**Definition 10** (Linear function).

**Definition 11** (Monomial and $n$-th root). Let $w = z^n$ be a monomial of degree $n \in \mathbb{N}$. Using the polar notation we see that $(r\angle\phi)^n = r^n\angle(n\phi)$. Because $r\angle\phi = r\angle(\phi + 2\pi)$ there cannot be a bijection between $w$ and $z$, if we want to define an inverse function $z = \sqrt[n]{w}$ we get many values with the form

$$z_k = \sqrt[n]{r}\angle(\phi + k2\pi)/n \qquad 0 \leq k < n$$

This fact implies that in general for $z, u \in \mathbb{C}$ $\sqrt[n]{zu} \neq \sqrt[n]{z}\sqrt[n]{u}$, as the relationship holds only for *some* values of $\sqrt[n]{z}$ and $\sqrt[n]{u}$.

**Theorem 2** (Roots of a polynomial). Every complex polynomial of degree $n$ has always $n$ roots in $\mathbb{C}$.

**Theorem 3.** Every complex polynomial of degree $n$ with coefficients can be *uniquely* rewritten in term of its roots.

$$P(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{k=0}^{n} (z - z_k)$$

**Theorem 4** (Polynomal with real coefficients). The roots of a polynomial with real coefficients of degree $n$, always come in conjugate complex pairs of $r$ and $\overline{r}$. That is because

$$(z - r)(z - \overline{r}) = z^2 - 2\operatorname{Re}(r)z + |z|^2$$

**Lemma 7.** From the previous theorem follows that a polynomial with real coefficients of *odd* degree, has *always* at least one real solution because $r \in \mathbb{R} \iff r = \overline{r}$.

**Theorem 5.** All roots of a polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ are inside of the open disk centered at the origin of radius $\sum_{k=0}^{n} |a_k/a_n|$.

**Theorem 6** (Cardano's cubic formula).

**Definition 12** (Exponential). If $z$ is a complex number we define the exponential function $e^z$ by its convergent power series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

**Theorem 7** (Euler's formula). By setting the argument of the exponential function to $jt$ for some $t \in \mathbb{R}$ we can reorder the power series to be a sum of the power series of $j\sin$ and $\cos$, and thus define

$$e^{jt} = \cos t + j \sin t = \operatorname{cjs} t = 1\angle t$$

**Lemma 8** (Rules for exponents). Let $a, b \in \mathbb{C}$ and $k \in \mathbb{Z}$, we can show that

$$e^a e^b = e^{a+b} \quad e^a/e^b = e^{a-b} \quad (e^a)^k = e^{ak}$$

**Definition 13** (Trigonometric functions). When $z$ is a complex number we define

$$\cos z = \frac{e^{jz} + e^{-jz}}{2} \qquad \sin z = \frac{e^{jz} - e^{-jz}}{2j}$$

like the (real) hyperbolic trigonometric functions

$$\cosh z = \left(e^z + e^{-z}\right)/2 \qquad \sinh z = \left(e^z - e^{-z}\right)/2$$

Notice that the sinus function is point symmetric to $\pi/2$, because $\sin(\pi/2 - z) = \sin(\pi/2 + z)$.

**Lemma 9** (Some trigonometric identities). Let $x, a, b \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$

$$\sin(x + \pi/2) = \cos(x) \qquad \cos(x - \pi/2) = \sin(x)$$
$$\sinh(jx) = j\sin(x) \qquad \cosh(jx) = \cos(x)$$
$$\sin(a + jb) = \sin(a)\cosh(b) + j\cos(a)\sinh(b)$$
$$\cos(a + jb) = \cos(a)\cosh(b) + j\sin(a)\sinh(b)$$
$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$
$$2\sin(\alpha)\cos(\beta) = \sin(\alpha - \beta) + \sin(\alpha + \beta)$$

**Lemma 10** (Superposition of sinuses). Let $s(t) = A\sin(\omega t + \varphi)$ be a sinusoidal wave. We can rewrite $s$ in complex form with

$$s(t) = \operatorname{Im}\left(Ae^{j(\omega t + \varphi)}\right) = \operatorname{Im} Ae^{j\varphi} \cdot e^{j\omega t}$$

If we now wish to sum $N$ sinusoids with the same frequency $\omega$, the resulting sinusoid $A\sin(\omega t + \varphi)$ has

$$A = \left|\sum_{n=1}^{N} A_n e^{j\varphi_n}\right| \qquad \varphi = \arg \sum_{n=1}^{N} A_n e^{j\varphi_n}$$

**Definition 14** (Logarithm). Because $w = e^z$ defined from $\mathbb{C} \to \mathbb{C}$ is not a bijection ($e^{z+2\pi j} = e^z$), unless we restrict the imaginary part of the domain to $(\pi, \pi]$, we get only an equivalence relationship because

$$\ln\left[|w|e^{j(\phi + k2\pi)}\right] = \ln|w| + j(\phi + k2\pi)$$

where $k \in \mathbb{Z}$. Similarly for $z, w \in \mathbb{C}$

$$\ln(w) \equiv z \pmod{2\pi j}$$
$$\ln(w^k) \equiv k\ln(w) \pmod{2\pi j}$$
$$\ln(zw) \equiv \ln(z) + \ln(w) \pmod{2\pi j}$$
$$\ln(z/w) \equiv \ln(z) - \ln(w) \pmod{2\pi j}$$

**Lemma 11** (General exponentiation). So far we have only exponentiation for an exponent $k \in \mathbb{Z}$, by adding $m \in \mathbb{N}$ we can define the quotient $k/m \in \mathbb{Q}$ that together with $z \in \mathbb{C}$ gives

$$\begin{aligned} z^{k/m} &= e^{\ln(z)k/m} \\ &= \exp\left((\ln|z| + j(\arg z + 2\pi n))k/m\right) \\ &= \exp\left(\ln|z| \cdot k/m\right)\exp((\arg z + 2\pi n)jk/m) \\ &= |z|^{k/m}\exp\left((\arg z + 2\pi n)jk/m\right) = \sqrt[m]{z^k} \end{aligned}$$

like in the reals, except that we have $m$ values because of the $m$-th root. If we let $w \in \mathbb{C}$ the expression $z^w$ cannot be equal to an unique value because

$$\begin{aligned} z^w = e^{w\ln z} &= \exp\left(w(\ln|z| + j\arg z + 2\pi nj)\right) \\ &= e^{w(\ln|z| + j\arg z)}e^{w2\pi nj} \end{aligned}$$

instead it is said to be *multivalued*. This means that there are no general exponentiation rules.

## 3 Fourier Series

**Definition 15** (Real trigonometric polynomial)**.** Let $\omega = 2\pi/T \in \mathbb{R}$ and $A_n, B_n$ be sequences in $\mathbb{R}$. We define a *real trigonometric polynomial* of degree $N$ to be

$$\tau_N(t) = \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos(n\omega t) + B_n \sin(n\omega t)$$

**Lemma 12** (Orthogonality of the basis functions)**.** Let $m, n \in \mathbb{N}_0$

$$\int_0^T \cos(m\omega t)\cos(n\omega t) = \begin{cases} T & m = n = 0 \\ T/2 & m = n > 0 \\ 0 & m \neq n \end{cases}$$

$$\int_0^T \sin(m\omega t)\sin(n\omega t) = \begin{cases} T/2 & m = n \wedge n \neq 0 \\ 0 & m \neq n \\ 0 & m = 0 \vee n = 0 \end{cases}$$

$$\int_0^T \cos(m\omega t)\sin(n\omega t) = 0$$

**Definition 16.** We denote with $\Omega$ the space of real valued, $T$-periodic, piecewise continuous functions, that have only a finite number of discontinuities, in which both the right and left limit exist, within the interval $[0, T)$.

**Theorem 8** (Fourier coefficients)**.** For any $f \in \Omega$ we can now define the *Fourier coefficients*

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,\mathrm{d}t \qquad a_0 = \frac{2}{T}\int_0^T f(t)\,\mathrm{d}t$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,\mathrm{d}t \qquad b_0 = 0$$

Worth noting are the special cases when $n = 0$.

**Definition 17** (Fourier Polynomial)**.** We can now use the Fourier coefficients as sequences for a trigonometric polynomial to obtain a *Fourier Polynomial*

$$S_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

**Lemma 13.** A trigonometric polynomial has the smallest distance (by the $L^2$ metric) from a function $f \in \Omega$, iff $A_n = a_n$ and $B_n = b_n$, in other words iff it is a Fourier Polynomial.

**Definition 18** (Fourier Series)**.** We can finally define the *Fourier Series* to be the infinite Fourier Polynomial, by letting $N \to \infty$

$$S(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

**Theorem 9** (Fourier coefficients of even and odd functions)**.** Recall that a function is said to be *even* if $f(-x) = f(x)$ or *odd* if $f(-x) = -f(x)$. We can show that if a function is

- even, then $b_n = 0$ for all $n$, and

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega t)\,\mathrm{d}t$$

- odd, then $a_n = 0$ for all $n$, and

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega t)\,\mathrm{d}t$$

**Lemma 14** (Linearity of Fourier coefficients)**.** Recall that linearity means $L(\mu x + \lambda y) = \mu L(x) + \lambda L(y)$. We then let $f, g \in \Omega$ be functions with Fourier series and $h = \mu f + \lambda g$ where $\mu, \lambda \in \mathbb{R}$ are constants. By denoting with $a_n^{(f)}$ the Fourier coefficient $a_n$ of the function $f$, and similarly with $b_n^{(f)}$, it is easily shown that

$$a_n^{(h)} = \mu a_n^{(f)} + \lambda a_n^{(g)} \qquad b_n^{(h)} = \mu b_n^{(f)} + \lambda b_n^{(g)}$$

**Lemma 15** (Fourier coefficients after time dilation)**.** Let $f \in \Omega$ be a function with a Fourier Series and $g(t) = f(rt)$ with $0 \neq r \in \mathbb{R}$. It follows that $a_n^{(g)} = a_n^{(f)}$ and $b_n^{(g)} = \operatorname{sgn}(r) \cdot b_n^{(f)}$.

**Lemma 16** (Fourier coefficients after time translation)**.** Let $f \in \Omega$ be a function with a Fourier Series and $g(t) = f(t + \tau)$ with $\tau \in \mathbb{R}$. It follows that

$$\begin{aligned} a_n^{(g)} &= \cos(n\omega\tau) \cdot a_n^{(f)} + \sin(n\omega\tau) \cdot b_n^{(f)} && n \geq 0 \\ b_n^{(g)} &= -\sin(n\omega\tau) \cdot a_n^{(f)} + \cos(n\omega\tau) \cdot b_n^{(f)} && n > 0 \end{aligned}$$

**Theorem 10** (Fourier theorem)**.** For any $f \in \Omega$ the Fourier series of $f$ converges in $L^2$ metric to $f$.

$$\lim_{N\to\infty} \left\| \frac{a_0}{2} + \sum_{n=0}^{N} a_n \cos(n\omega t) + b_n \sin(n\omega t) - f(t) \right\| = 0$$

**Theorem 11** (Plancherel Parselval theorem)**.** Let $f \in \Omega$ with a Fourier Series with coefficients $a_n$ and $b_n$.

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) \leqq \frac{2}{T}\int_0^T |f(t)|^2\,\mathrm{d}t = \|f\|^2$$

**Theorem 12.** Both sequences $a_n, b_n$ for the Fourier coefficients of a function $f \in \Omega$ converge to zero.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,\mathrm{d}t = 0$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,\mathrm{d}t = 0$$

**Theorem 13** (Rate of convergence of Fourier coefficients)**.** If $f$ is a $T$-periodic, $(m-2)$ times differentiable, continuous function. And if its $(m-1)$-th derivative is pieceweise monotonous and $\in \Omega$, then there exists a constant $c \in \mathbb{R}$ such that

$$|a_n| \leq \frac{c}{n^m} \qquad |b_n| \leq \frac{c}{n^m} \qquad m, n \in \mathbb{N}$$

**Theorem 14** (Integration and differentiation of the Fourier series)**.** It is possible to show from the previous theorem (and others before) that when $m \geq 2$ the Fouriers converges *uniformly*. This means that it is possible to integrate or differentiate the series term by term.

$$f'(t) = \sum_{n=1}^{\infty} b_n n\omega \cos(n\omega t) - a_n n\omega \sin(n\omega t)$$

and

$$\int_0^t f(\tau)\,\mathrm{d}\tau = \left( \sum_{n=1}^{\infty} \frac{b_n}{n\omega} \right) + \frac{a_0}{2} t + \left( \sum_{n=1}^{\infty} \frac{a_n}{n\omega} \sin(n\omega t) - \frac{b_n}{n\omega} \cos(n\omega t) \right)$$

**Theorem 15** (Dirichlet pointwise convergence)**.** Let $f \in \Omega$, then it is known that its Fourier series converges to

$$\lim_{\epsilon \to 0} \frac{f(t-\epsilon) + f(t+\epsilon)}{2}$$

for every $t$, where the left and right derivative $f'(t-\epsilon)$, $f'(t+\epsilon)$, with $\epsilon \to 0$, exist.

In the special case where $f$ is continuous at $t$, and the derivatives exist, there the Fourier series converges exactly to $f(t)$, i.e. the value of the function at $t$.

**Definition 19** (Complex representation of the Fourier coefficients)**.** By letting $n \in \mathbb{Z}$ and

$$c_n = \overline{c_{-n}} = \frac{a_n - jb_n}{2} = \frac{1}{T} \int_0^T f(t) e^{-jn\omega t}\,\mathrm{d}t$$

using a notational trick for negative indices. We can compactly write a Fourier series or polynomial as

$$S(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

**Theorem 16** (Complex Fourier coefficients of even and odd functions)**.** By the definition of $c_n$ and the previous similar theorem for the real coefficients, it is clear that when a function $f \in \Omega$ is *even*, then $\mathrm{Im}(c_k) = 0$, whereas when $f$ is *odd* $\mathrm{Re}(c_k) = 0$ $(k \in \mathbb{Z})$.

**Theorem 17** (Complex Fourier coeffients after time translation)**.** Similarly to the previous theorem, we can now compactly write that if $f \in \Omega$ has a Fourier series with coefficients $c_k^{(f)}$, and $g(t) = f(t+\tau)$, then

$$c_k^{(g)} = e^{jk\omega\tau} c_k^{(f)} \qquad k \in \mathbb{Z}$$

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