SigSys Zusammenfassung

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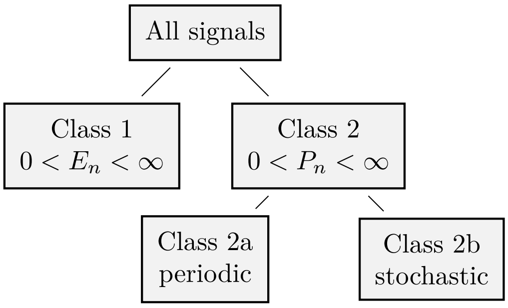
Part I
Signals and Systems

1 Signals

1.1 Classification



1.2 Properties

For class 2b signals the formula for class 2a signals can used by taking $\lim_{T\to\infty} f_{2\text{a}}(T)$ (if the limits exists). The notation $\int_T$ is short for an integral from $-T/2$ to $T/2$.

Characteristic	**Symbol and formula**
Class 1 Signals	
Normalized energy	$E_n = \lim_{T\to\infty} \int_T \lvert x\rvert^2 \, dt$
Class 2a Signals	
Normalized power	$P_n = \lim_{T\to\infty} \frac{1}{T} \int_T \lvert x\rvert^2 \, dt$
Linear mean	$X_0 = \frac{1}{T} \int_T x \, dt$
Mean square	$X^2 = \frac{1}{T} \int_T x^2 \, dt$
n-th order mean	$X^n = \frac{1}{T} \int_T x^n \, dt$
Rectified value	$\lvert\bar{X}\rvert = \frac{1}{T} \int_T \lvert x\rvert \, dt$
Variance	$\sigma^2 = \frac{1}{T} \int_T (x - X_0)^2 \, dt$
	$= X^2 - X_0$
Root mean square	$X_{\text{rms}} = \sqrt{X^2}$

1.3 Correlation

Autocorrelation The *autocorrelation* is a measure for how much a signal is coherent, i.e. how similar it is to itself. For class 1 signals the autocorrelation is

$$\varphi_{xx}(\tau) = \lim_{T\to\infty} \int_T x(t)x(t-\tau)\, dt,$$

whereas for class 2a and 2b signals

$$\varphi_{xx}(\tau) = \frac{1}{T} \int_T x(t)x(t-\tau)\, dt \quad \text{(2a)},$$

$$\varphi_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T x(t)x(t-\tau)\, dt \quad \text{(2b)}.$$

Properties of φ_{xx}:

- $\varphi_{xx}(0) = X^2 = (X_0)^2 + \sigma^2$
- $\varphi_{xx}(0) \geq \lvert\varphi_{xx}(\tau)\rvert$
- $\varphi_{xx}(\tau) \geq (X_0)^2 - \sigma^2$
- $\varphi_{xx}(\tau) = \varphi_{xx}(\tau + nT)$ (periodic)
- $\varphi_{xx}(\tau) = \varphi_{xx}(-\tau)$ (even, symmetric)

The Fourier transform of the autocorrelation $\Phi_{xx}(j\omega) = \mathcal{F}\,\varphi_{xx}(t)$ is called *energy spectral density* (ESD) for class 1 signals or *power spectral density* (PSD) for class 2 signals.

Cross correlation The *cross correlation* measures the similarity of two different signals x and y. For class 1 signals

$$\varphi_{xy}(\tau) = \lim_{T\to\infty} \int_T x(t)y(t-\tau)\, dt.$$

Similarly for class 2a and 2b signals

$$\varphi_{xy}(\tau) = \frac{1}{T} \int_T x(t)y(t-\tau)\, dt \quad \text{(2a)},$$

$$\varphi_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T x(t)y(t-\tau)\, dt \quad \text{(2b)}.$$

Properties of φ_{xy}:

- For signals with different frequencies φ_{xy} is always 0.
- For stochastic signals $\varphi_{xy} = 0$

1.4 Amplitude density

The amplitude density is the probability that a signal has a certain amplitude during a time interval T.

$$p(a) = \frac{1}{T}\frac{dt}{dx} \in [0,1]$$

2 LTI systems

2.1 Properties

Let $\mathcal{S}$ denote a system.

Property	Meaning
static $\leftrightarrow$ dynamic	Static means that it is memoryless (in the statistical sense), whereas dynamic has memory. Static systems depend only on the input u, dynamic systems on du/dt or $\int u\,dt$.
causal $\leftrightarrow$ acausal	Causal systems use only informations from the past, i.e. $h(t<0)=0$. Real systems are always causal.
linear $\leftrightarrow$ nonlinear	The output of a linear system does not have new frequency that were not in the input. For linear system the superposition principle is valid: $\mathcal{S}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \mathcal{S} x_1 + \alpha_2 \mathcal{S} x_2$.
time invariant $\leftrightarrow$ time variant	Time invariant systems do not depend on time, but for ex. only on time differences.
SISO, MIMO	Single input single output, multiple input multiple output.
BIBO	Bounded input bounded output, i.e. there are some A, B such that $\lvert x\rvert < A$ and $\lvert y\rvert < B$ for all t, equivalently $\int_{\mathbb{R}} \lvert h\rvert\,dt < \infty$.

2.2 Time domain description

A general LTI system with input x and output y is described in the time domain with a linear differential equation of the form

$$\sum_{i=0}^{n} a_i y^{(i)} = \sum_{k=0}^{m} b_k x^{(k)}.$$

2.3 Impulse response

2.4 Transfer function

By taking the Laplace transform of the differential equation of the system a and assuming all initial conditions to be zero, we obtain

$$Y \sum_{i=0}^{n} a_i s^i = X \sum_{k=0}^{m} b_k s^k,$$

where Y and X are the Laplace transform of y and x respectively. We then define the *transfer function* to be the ratio $H = Y/X$, or

$$H(s) = \frac{\sum_{k=0}^{m} b_k s^k}{\sum_{i=0}^{n} a_i s^i} = \frac{\prod_{k=0}^{m} s - z_k}{\prod_{i=0}^{n} s - p_i},$$

since polynomials can be expressed in terms of their roots. We say the roots of Y are *zeroes* and those of X *poles*, because of how they appear in the complex plane of H.

2.5 Frequency response

2.6 Stability

Let $\mathcal{S}$ be a system with impulse response $h(t)$ and transfer function $H(s)$.

Stable	All poles are on the LHP[1].
Marginally stable	There are no poles in the RHP but a simple pole on the j-axis.
Instable	There are poles in the RHP or poles of hider order on the j-axis.

2.7 Distortion

For a periodic signal the Fourier transform is a bunch of weighted Dirac deltas (or a Fourier series), i.e.

$$\mathcal{F}\{f\} = \sum_i d_i \delta(\omega - \omega_i).$$

The spectrum of a sinusoidal signal of frequency ω_1 is only one weighted delta $d_1\delta(\omega - \omega_1)$. When a system introduces a *nonlinear* distortion, with a clean sine input new higher harmonics are found in the output.

To measure the distortion of a signal in the English literature there is the *total harmonic distortion* (THD) defined as

$$\mathrm{THD} = \frac{1}{d_1}\sqrt{\sum_{i=2}^{n} d_i^2}.$$

In the German literature there is the distortion factor (*Klirrfaktor*, always between 0 and 1)

$$k = \sqrt{\frac{d_2^2 + d_3^2 + \cdots + d_n^2}{d_1^2 + d_2^2 + \cdots + d_n^2}}.$$

Both are usually given in percent (%) and are related with

$$(\mathrm{THD})^2 = \frac{k^2}{1-k^2},$$

thus $\mathrm{THD} \geq k$.

2.8 Stochastic inputs

3 State space representation

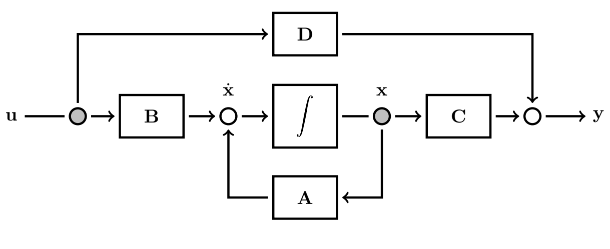


Figure 1: Diagram of a LTI MIMO system with vector variables.

Symbol	Size	Name
$\mathbf{x}$	n	State vector
$\mathbf{u}$	m	Output vector
$\mathbf{y}$	k	Output vector
$\mathbf{A}$	$n \times n$	System matrix
$\mathbf{B}$	$m \times n$	Input matrix
$\mathbf{C}$	$n \times k$	Output matrix
$\mathbf{D}$	$k \times m$	Feed forward matrix

Table 1: Matrices for a state space representation

A system described by a system of linear differential equations of n-th order, can be equivalently be described by n first order differential equations. Which can be compactly written in matrix form as

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}.\end{aligned}$$

If the system is time *variant* the matrices are functions of time.

3.1 Canonical representations

3.1.1 Controllable form

3.1.2 Observable form

3.1.3 Diagonalized or Jordan form

The Jordan form diagonalizes the $\mathbf{A}$ matrix. Thus we need to solve the eigenvalue problem $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, which can be done by setting $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, and solving the characteristic polynomial. The eigenvectors are obtained by plugging the λ values back into $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, and solving an overdetermined system of equations. Tip: for a 2×2 matrix $\mathbf{A}$ the eigenvalues can be quickly calculated with

$$m = \frac{1}{2}\operatorname{tr}\mathbf{A} = \frac{a+d}{2}, \quad p = \det\mathbf{A} = ad - bc$$

and then $\lambda_{1,2} = m \pm \sqrt{m^2 - p}$.

The transformation to the eigenbasis $\mathbf{T}$, obtained by using the eigenvector as columns of a matrix $\mathbf{T} = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}$, is then used to compute

$$\begin{aligned}\hat{\mathbf{A}} &= \mathbf{TAT^{-1}} & \hat{\mathbf{B}} &= \mathbf{TB} \\ \hat{\mathbf{C}} &= \mathbf{CT^{-1}} & \hat{\mathbf{D}} &= \mathbf{D}.\end{aligned}$$

In this form the system is described with n decoupled states ξ_i with the equations $\dot{\boldsymbol{\xi}} = \hat{\mathbf{A}}\boldsymbol{\xi} + \hat{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \hat{\mathbf{C}}\boldsymbol{\xi} + \hat{\mathbf{D}}\mathbf{u}$.

3.2 Stability

If *all* eigenvalues λ are not zero and have a positive real part the system is asymptotically *stable*. If *all* eigenvalues are not zero but *at least one* has a negative real part the system is *unstable*.

3.3 Controllability

The state controllability condition implies that it is possible — by admissible inputs — to steer the states from any initial value to any final value within some finite time window. A LTI state space model is controllable iff the matrix

$$\mathbf{Q} = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B}\cdots\mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$$

has $\operatorname{rank}\mathbf{Q} = n$. For a SISO system, if all components of the vector $\hat{\mathbf{B}}$ are not zero, then the system is controllable.

3.4 Observability

Observability is a measure for how well internal states of a system can be inferred by knowledge of its external outputs. A LTI state space mode is observable iff the matrix

$$\mathbf{Q}^t = \begin{bmatrix}\mathbf{C} & \mathbf{CA} & \cdots & \mathbf{CA}^{n-1}\end{bmatrix}$$

has $\operatorname{rank}\mathbf{Q} = n$. For a SISO system it is also possible to infer observability from the diagonalized form: if all elements of the $\hat{\mathbf{C}}$ are not zero, then the system is observable.

3.5 Solutions in time domain

3.6 Solutions in the s-domain

By taking the Laplace transform of the system of differential equations we obtain

$$\begin{aligned}s\mathbf{X} - \mathbf{x}(0) &= \mathbf{AX} + \mathbf{BU} \\ \mathbf{Y} &= \mathbf{CX} + \mathbf{DU}.\end{aligned}$$

The first equation can be solved for $\mathbf{X}$ giving

$$\mathbf{X} = (s\mathbf{I} - A)^{-1}\left(\mathbf{x}(0) + \mathbf{BU}\right).$$

Substituting in the second equation results in

$$\mathbf{Y} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\left(\mathbf{x}(0) + \mathbf{BU}\right) + \mathbf{DU}.$$

Assuming that the initial conditions $\mathbf{x}(0) = \mathbf{0}$, then

$$\mathbf{Y} = \left(\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)\mathbf{U}.$$

from which can define the *transfer matrix* $\mathbf{H}$ to be the matrix that takes $\mathbf{Y}$ to $\mathbf{U}$, i.e.

$$\mathbf{H} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D},$$

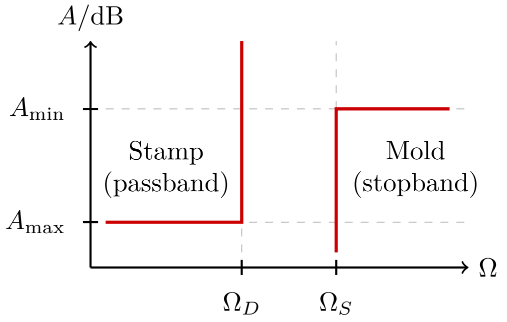
that we can use to compute $y = \mathcal{L}^{-1}\{\mathbf{H}\,\mathcal{L}\,u\}$.

In the special case of a SISO system the transfer matrix $\mathbf{H}$ is one dimensional and exactly equal to the transfer function H.

4 Filters

4.1 Normalized Frequency

For this section we will always use a normalized frequency $\Omega = \omega/\omega_r$ for some reference frequency ω_r. For TP and HP filters $\omega_r = \omega_D$ (cut-off frequency), whereas for BP and BS $\omega_r = \omega_m$ (frequency in the middle of the band).



4.2 LPF Approximations

The approximations of ideal low pass filters generally have (with some exceptions) the form

$$|H(j\Omega)|^2 = H(j\Omega) \cdot H^*(j\Omega) = \frac{1}{1 + K(\Omega^2)},$$

where K is the so called *charcteristic function.* For a nicer notation we will define the *attenuation* function

$$A(\Omega) = 10 \log \left(|H(j\Omega)|^{-2} \right), \quad [A] = \mathrm{dB}.$$

With that, ideally wish to have an approximation that satisfies the following requirements:

- $A(\Omega = 0) = 1 = 0$ dB
- $A(\Omega = 1) = 1/\sqrt{2} \approx -3$ dB
- $A(\Omega \to \infty) = 0$

Critically damped filter

Butterworth Let $K(\Omega^2) = \Omega^{2n}$, thus

$$A(\Omega) = 10 \log \left(1 + \Omega^{2n} \right).$$

To find the order of the filter given two parameters the formula is

$$n = \left\lceil \frac{1}{2} \log \left(\frac{10^{A_{\min}/10} - 1}{10^{A_{\max}/10} - 1} - \frac{\Omega_S}{\Omega_D} \right) \right\rceil.$$

Chebyshev I Let $K(\Omega^2) = e^2 C_n^2(\Omega)$, so

$$A(\Omega) = 10 \log \left(1 + e^2 C_n^2(\Omega) \right),$$

where $C_n = \cos(n \arccos(\Omega))$ for $|\Omega| \leq 1$ (in the passband), and when $|\Omega| > 1$ (in the stopband) $C_n = \cosh(n \operatorname{arccosh}(\Omega))$, is a so called Chebyshev polynomial of n-th order. The ripple factor e is a parameter, *not the natural number (2.71...).* Chebyshev polynomials can be computed recursively with the formula

$$C_n = 2\Omega C_{n-1} - C_{n-2},$$

and knowing that $C_1 = \Omega$ and $C_2 = 2\Omega^2 - 1$.

The idea is that in the passband the attenuation is periodic and stays more or less constant, and in the stopband the function is no longer periodic and damps the frequencies. To find the ripple factor e given an $A_{\max}$

$$e = \sqrt{10^{A_{\max}/10} - 1},$$

and to find the order given two parameters

$$n = \left\lceil \frac{\operatorname{arccosh} \sqrt{\frac{10^{A_{\min}/10} - 1}{10^{A_{\max}} - 1}}}{\operatorname{arccosh}(\Omega_S/\Omega_D)} \right\rceil.$$

Chebyshev II Also known as *inverse* Chebyshev because $K(\Omega^2) = 1/e^2 C_n^2(1/\Omega)$.

Cauer

Part II
Mathematics