SigSys Zusammenfassung

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Contents

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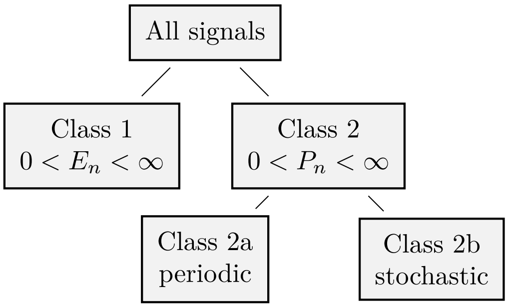
Part I

Signals and Systems

1 Signals

1.1 Classification



1.2 Properties

For class 2b signals the formula for class 2a signals can used by taking $\lim_{T\to\infty} f_{2\text{a}}(T)$ (if the limits exists). The notation $\int_T$ is short for an integral from $-T/2$ to $T/2$.

Characteristic	**Symbol and formula**
Class 1 Signals	
Normalized energy	$E_n = \lim_{T\to\infty} \int_T \lvert x\rvert^2 \, dt$
Class 2a Signals	
Normalized power	$P_n = \lim_{T\to\infty} \frac{1}{T} \int_T \lvert x\rvert^2 \, dt$
Linear mean	$X_0 = \frac{1}{T} \int_T x \, dt$
Mean square	$X^2 = \frac{1}{T} \int_T x^2 \, dt$
n-th order mean	$X^n = \frac{1}{T} \int_T x^n \, dt$
Rectified value	$\lvert\bar{X}\rvert = \frac{1}{T} \int_T \lvert x\rvert \, dt$
Variance	$\sigma^2 = \frac{1}{T} \int_T (x - X_0)^2 \, dt$
	$= X^2 - X_0$
Root mean square	$X_{\text{rms}} = \sqrt{X^2}$

1.3 Correlation

Autocorrelation The *autocorrelation* is a measure for how much a signal is coherent, i.e. how similar it is to itself. For class 1 signals the autocorrelation is

$$\varphi_{xx}(\tau) = \lim_{T\to\infty} \int_T x(t)x(t-\tau) \, dt,$$

whereas for class 2a and 2b signals

$$\varphi_{xx}(\tau) = \frac{1}{T} \int_T x(t)x(t-\tau) \, dt \quad \text{(2a)},$$
$$\varphi_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T x(t)x(t-\tau) \, dt \quad \text{(2b)}.$$

Properties of φ_{xx}:

- $\varphi_{xx}(0) = X^2 = (X_0)^2 + \sigma^2$
- $\varphi_{xx}(0) \geq \lvert\varphi_{xx}(\tau)\rvert$
- $\varphi_{xx}(\tau) \geq (X_0)^2 - \sigma^2$
- $\varphi_{xx}(\tau) = \varphi_{xx}(\tau + nT)$ (periodic)
- $\varphi_{xx}(\tau) = \varphi_{xx}(-\tau)$ (even, symmetric)

The Fourier transform of the autocorrelation $\Phi_{xx}(j\omega) = \mathcal{F}\,\varphi_{xx}(t)$ is called *energy spectral density* (ESD) for class 1 signals or *power spectral density* (PSD) for class 2 signals.

Cross correlation The *cross correlation* measures the similarity of two different signals x and y. For class 1 signals

$$\varphi_{xy}(\tau) = \lim_{T\to\infty} \int_T x(t)y(t-\tau) \, dt.$$

Similarly for class 2a and 2b signals

$$\varphi_{xy}(\tau) = \frac{1}{T} \int_T x(t)y(t-\tau) \, dt \quad \text{(2a)},$$
$$\varphi_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T x(t)y(t-\tau) \, dt \quad \text{(2b)}.$$

Properties of φ_{xy}:

- For signals with different frequencies φ_{xy} is always 0.
- For stochastic signals $\varphi_{xy} = 0$

1.4 Amplitude density

The amplitude density is the probability that a signal has a certain amplitude during a time interval T.

$$p(a) = \frac{1}{T} \frac{dt}{dx} \in [0, 1]$$

2 LTI systems

2.1 Properties

Let $\mathcal{S}$ denote a system.

Property	Meaning
static $\leftrightarrow$ dynamic	Static means that it is memoryless (in the statistical sense), whereas dynamic has memory. Static systems depend only on the input u, dynamic systems on du/dt or $\int u\, dt$.
causal $\leftrightarrow$ acausal	Causal systems use only informations from the past, i.e. $h(t < 0) = 0$. Real systems are always causal.
linear $\leftrightarrow$ nonlinear	The output of a linear system does not have new frequency that were not in the input. For linear system the superposition principle is valid: $\mathcal{S}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \mathcal{S} x_1 + \alpha_2 \mathcal{S} x_2$.
time invariant $\leftrightarrow$ time variant	Time invariant systems do not depend on time, but for ex. only on time differences.
SISO, MIMO	Single input single output, multiple input multiple output.
BIBO	Bounded input bounded output, i.e. there are some A, B such that $\|x\| < A$ and $\|y\| < B$ for all t, equivalently $\int_{\mathbb{R}} \|h\|\, dt < \infty$.

2.2 Impulse response

2.3 Stability

Let $\mathcal{S}$ be a system with impulse response $h(t)$ and transfer function $H(s)$.

Stable	All poles are on the LHP[1].
Marginally stable	There are no poles in the RHP but a simple pole on the j-axis.
Instable	There are poles in the RHP or poles of hider order on the j-axis.

2.4 Distortion

2.5 Stochastic inputs

3 Frequency response of LTI systems

4 State space representation

A system described by a system of linear differential equations of n-th order, can be equivalently be described by n first order differential equations. Which can be compactly written in matrix form as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}.$$

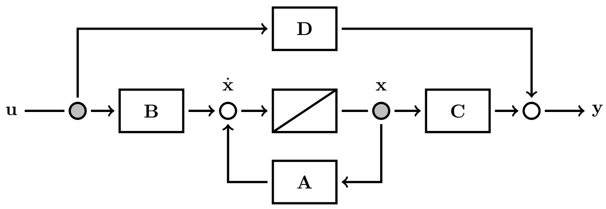


Figure 1: A LTI MIMO system.

Symbol	Size	Name
$\mathbf{x}$	n	State vector
$\mathbf{u}$	m	Output vector
$\mathbf{y}$	k	Output vector
$\mathbf{A}$	$n \times n$	System matrix
$\mathbf{B}$	$m \times n$	Input matrix
$\mathbf{C}$	$n \times k$	Output matrix
$\mathbf{D}$	$k \times m$	Feed forward matrix

Table 1: Matrices for a state space representation

If the system is time *variant* the matrices are functions of time.

4.1 Canonical representations

4.1.1 Controllable form

4.1.2 Observable form

4.1.3 Diagonalized or Jordan form

The Jordan form diagonalizes the $\mathbf{A}$ matrix. Thus we need to solve the eigenvalue problem $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, which can be done by setting $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, and solving the characteristic polynomial. The eigenvectors are obtained by plugging the λ values back into $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, and solving an overdetermined system of equations.

The transformation to the eigenbasis $\mathbf{T}$, obtained by using the eigenvector as columns of a matrix $\mathbf{T} = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}$, is then used to compute

$$\hat{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \qquad \hat{\mathbf{B}} = \mathbf{T}\mathbf{B}$$
$$\hat{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1} \qquad \hat{\mathbf{D}} = \mathbf{D}.$$

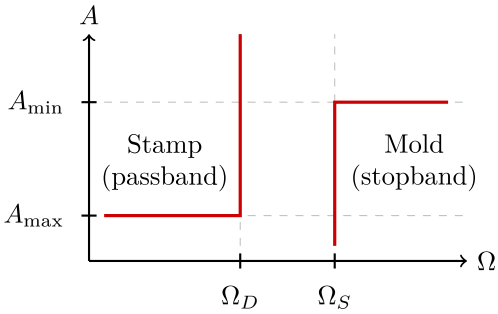
In this form the system is described with n decoupled states ξ_i with the equations $\dot{\boldsymbol{\xi}} = \hat{\mathbf{A}}\boldsymbol{\xi} + \hat{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \hat{\mathbf{C}}\boldsymbol{\xi} + \hat{\mathbf{D}}\mathbf{u}$.

4.2 Stability

If *all* eigenvalues λ are not zero and have a positive real part the system is asymptotically *stable*. If *all* eigenvalues are not zero but *at least one* has a negative real part the system is *unstable*.

4.3 Controllability

The state controllability condition implies that it is possible — by admissible inputs — to steer the states



from any initial value to any final value within some finite time window. A LTI state space model is controllable iff the matrix

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \cdots \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

has rank $\mathbf{Q} = n$. Or equivalently for a SISO system, if all components of the vector $\hat{\mathbf{C}}_i \neq 0$.

4.4 Observability

Observability is a measure for how well internal states of a system can be inferred by knowledge of its external outputs. A LTI state space mode is observable iff the matrix

$$\mathbf{Q}^t = \begin{bmatrix} \mathbf{C} & \mathbf{CA} & \cdots & \mathbf{CA}^{n-1} \end{bmatrix}$$

has rank $\mathbf{Q} = n$.

4.5 Solutions in time domain

4.6 Solutions in frequency domain

5 Filters

Part II
Mathematics